THE DISSERTATION COMMITTEE FOR HEE-JOON KIM CERTIFIES THAT THIS IS THE APPROVED VERSION OF THE FOLLOWING DISSERTATION:

AN EXPLORATORY STUDY OF TEACHERS' USE OF MATHEMATICAL KNOWLEDGE FOR TEACHING TO SUPPORT MATHEMATICAL ARGUMENTATION IN MIDDLE-GRADES CLASSROOMS

Committee:

Susan Empson, Supervisor

Nicole Shechtman

Vera Michalchik

Uri Treisman

Walter Stroup

An exploratory Study of Teachers' Use of Mathematical Knowledge for Teaching to Support Mathematical Argumentation in Middle-Grades Classrooms

by

Hee-Joon Kim, B.S.; M.S.Ed.

Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of phisolophy

The University of Texas at Austin

December 2011

Acknowledgements

First I thank my parents, Jungjik Kim and Eunyoung Oh, who inspired me the most. They have emphasized the importance of education for myself and others since I could remember. They taught me perseverance on a daily basis throughout my life. Their unconditional love and support has been my energy to survive in any circumstances.

Second, I want to thank my husband, Youngjin Hur, who always encouraged me that I could do it and I could do well. Without his continuous encouragement and support, I could have not finished my dissertation.

Third, I want to thank my supervisor, Susan Empson, who supported and encouraged me throughout probably the longest and slowest process with a dissertation that she has ever supervised. I came across a few barriers that slowed me down quite a bit. Without her patience and understanding, I could have definitely not come this far. She provided me both financial and intellectual support. She introduced me to the Bridging project that matched my interest where I got financial support. Also, She supported me with writing of my dissertation from little tips for writing to how to write a research paper. From brainstorming to completing writing, there was no place that her expertise did not influence. I deeply appreciate her.

And, I want to thank Nikki Shechtman, Jennifer Knudsen, and Vera Michalchik from SRI International, who taught me hands-on how to conduct research with a big project like Bridging from scratch. Their expertise helped me to develop my knowledge and skills to become a researcher. Their open-mindedness with my ideas encouraged and pushed me forward.

Also, I want to thank my committee members for investing their time reading and giving insightful comments on my dissertation. I feel grateful that they allowed me extra time to work on my dissertation.

I want to thank my colleagues too, Teresa and Bowyee from SRI, and Miae, Jungwon, Seokho, Soyeon, Luz, Li, and Teddy from UT. I loved talking mathematics with Teresa. Bowyee was always available whenever I needed help with the project. My colleagues from UT were the ones who kept me grounded and going. They constantly checked in with how I was doing and were supportive no matter what. They were the ones I could always lean on.

Finally, I want to thank the teachers who allowed their precious time and classrooms for me to come in and observe and to be interviewed. Without them, this study would have not been possible.

This research was supported by the National Science Foundation (NSF) under grand number 0455868 awarded to SRI International (PI: Jennifer Knudsen, Co-PIs: Susan Empson & Nicole Shechtman). The opinions expressed here are those of the author and do not reflect those of NSF.

AN EXPLORATORY STUDY OF TEACHERS' USE OF MATHEMATICAL KNOWLEDGE FOR TEACHING TO SUPPORT MATHEMATICAL ARGUMENTATION IN MIDDLE-GRADES CLASSROOMS

Hee-Joon Kim, Ph.D
The University of Texas at Austin, 2011

Supervisor: Susan Empson

Mathematical argumentation is fundamental to doing mathematics and developing new knowledge. Working from the view that mathematical argumentation is also integral to teaching and learning mathematics, this study investigated teachers' use of mathematical knowledge for teaching (MKT) to support student participation in mathematical argumentation. Classroom observations were made of three case-study teachers' implementation of a three-day curriculum unit on mathematical argumentation and supplemented with paper and pencil assessments of teachers' MKT. Teaching moves, or teachers' actions directed toward supporting argumentation, were identified as a unit of discourse in which MKT-in-action appeared. Teachers' MKT showed up in three types of teaching moves including: Revoicing by Reformulation, Responding to Student Difficulties, and Pressing for Generalization in Defining. MKT that was evident in these moves included knowledge of core information in argument, heuristic methods, and formulation of mathematical definition through and in argumentation. Findings highlight that supporting mathematical argumentation requires teachers to have a sophisticated understanding of the subject matter as well as how concepts develop through argumentation. Findings have limitations in understanding complex teaching practices by considering MKT as a single factor. The study has implications on teacher learning and MKT assessments.

Table of Contents

List of Tables x

List of Figures xi

CHAPTER ONE RATIONALE 1

Background of the Study 1

Statement of the Problem and Purpose of the Study 10

Research Questions and Method 11

Significance of the Study 12

CHAPTER TWO CONCEPTUAL FRAMEWORK 14

Overview 14

Mathematical Argumentation and Proof 15

A Theoretical Model for Classroom Mathematical Argumentation 23

Teaching Moves 33

Mathematical Knowledge for Teaching 39

MKT for Mathematical Argumentation 43

CHAPTER THREE METHODOLOGY 47

Design of Study 47

Context 48

Participants 50

Data Collection Procedures 54

Data Analysis Procedures 58

CHAPTER FOUR FINDINGS 66

Overview of Argumentation 66

Occurrence of Argument and Argumentation episode 67

Nature of Justification within Argumentation Episodes 71

Overview of Teaching Moves and MKT 77

Revoicing by Reformulation 79

Revoicing: Reformulate to Highlight Core Ideas 80

Revoicing: Reformulate to Introduce New Information 86

Responding to Difficulties 94

Responding to Difficulties: Decompose into Sub-Arguments to Simplify 96

Responding to Difficulties: Use Counter-Arguments to Challenge 102

Pressing for Generalization in Defining 115

Pressing for Generalization in Defining: To Justify 118

Pressing for Generalization in Defining: To Provide 125

Summary 135

CHAPTER FIVE DISCUSSION 139

Summary and Implications 139

Limitations 145

Directions for Future Research 147

Conclusion 148

APPENDIX A. Example Items of MKT Assessment...151

APPENDIX B. Similar Rectangles 153

References..157

List of Tables

Table 3.1 Mathematical Conceptual Framework for PD, Student Curriculum, and Assessment 49

Table 3.2 Teacher's MKT Reflected in Open-Ended Items 51

Table 3.3 Characteristics of Case Study Participants and Their School 53

Table 3.4 Mathematical Goals for Each Activity in the Curriculum 55

Table 3.5 Post-lesson Interview Protocol 57

Table 3.6 Sample Codes 61

Table 3.7 Categories of Justification 62

Table 3.8 Codes for Teaching Moves 64

Table 4.1 Argument and Argumentation Episode 68

Table 4.2 Teaching Moves and Their Purposes 78

Table 4.3 Revoicing By Reformulation 94

Table 4.4 Responding to Difficulties 115

Table 4.5 Pressing for Generalization 134

Table 4.6 Summary of Findings 136

List of Figures

Figure 2.1. A basic form of argument 24
Figure 2.2. A basic form of mathematical argument showing two elements of justification 25
Figure 2.3. A basic process of argumentation 27
Figure 2.4. A theoretical model for classroom mathematical argumentation 28
Figure 4.1. Structure of Jason's Argument 72
Figure 4.2. Structure of an hypothetical argument 122

CHAPTER ONE: RATIONALE

BACKGROUND OF THE STUDY

Mathematical Argumentation

Mathematical argumentation is important to include in the K-12 curriculum because, from a historical perspective, mathematical argumentation is essential to doing mathematics (Hersh, 1997; Lampert, 1990; Schoenfeld, 1994). Mathematical argumentation is generally considered the core activity through which mathematical ideas are advanced and validated in a community. Also, from a pedagogical perspective, it enhances the development of students as mathematical thinkers. Despite its essential role in the professional community, mathematical argumentation has not been a central part of school mathematics. Instead, it has been marginalized as a topic of study in two-column format in high school geometry.

However, recently there has been emphasis placed on the importance of mathematical argumentation for school mathematics in policy documents. The Principles and Standards for School Mathematics (NCTM, 2000) recommends students in K-12 grades "recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof" (p. 56). A recently published Common Core State Standards for Mathematics (2010) also recommends students develop proficiency in constructing arguments using previously established statements and critiquing others' arguments.

Furthermore, the National Research Council (2001) views "the capacity to think logically about the relationships among concepts and situations" (p. 129) as part of mathematical proficiency, called adaptive reasoning. A key aspect of adaptive thinkers is the ability to justify their thinking to themselves and to convince others using mathematical reasons rather than relying on authority figures such as teachers. In turn, a productive disposition can be developed through making sense of mathematics by participating in a mathematical argument with peers with the support of a teacher.

There exist two distinctive perspectives on the nature and role of argumentation in mathematics. In the first perspective, mathematical argumentation is viewed as an activity that involved generating ideas by formal logic using previously established statements such as axioms, definitions, or theorems, which is called proof[1]. I refer to this view as the formal perspective. Often, its role has been limited to the verification of the truth or falsity of conjecture. This perspective has widely influenced the K-12 curriculum and classroom practices of mathematical argumentation. In light of students' developmental capability to understand formal logic and symbols, mathematical argumentation as proof typically has been introduced as a topic in high school geometry and classroom practices have focused on writing two-column proofs. This view of

---

[1] Hereafter, the term "proof" is used to indicate a formal perspective on justification that is based on formal logic, that usually appears in professional mathematics journals or high school geometry textbooks, in written form.

argumentation as formal proof has prevented students from learning mathematical argumentation as an integral part of doing mathematics.

A more recent broader perspective views mathematical argumentation as involving heuristic reasoning to create and advance mathematical ideas, while considering deductive proof as the final product (Lakatos, 1976; Polya, 1945/2004; Thurston, 1998; Hersh, 1997). I refer to this view as the emergent perspective. It involves both heuristic and deductive reasoning in discovery along with verification (de Villiers, 1990; Hana, 1991; Hersh, 1990; Schoenfeld, 1994; Thurston, 1998). What has been missing in the current practices is the opportunity for students to learn mathematical argumentation as an integral part of doing mathematics. Mathematical argumentation is not just a topic to learn, but it is a thinking process to create and advance mathematical ideas.

Moreover, the emergent perspective views mathematical argumentation and proof as social and context-dependent. Thurston (1998) asserts that evaluation of an argument depends on specific community that is involved in argument. Members of a community such as professional mathematics or a middle-grades classroom have a responsibility to convince each other by providing mathematical reasons. This perspective implies that in a classroom, students would be expected to decide whether ideas are mathematically convincing or not based on the knowledge that is shared among them. It does not mean that correct arguments in one community can be incorrect in other community. Instead, an argument that is accepted as valid in one community may not be validated in another depending on what has been shared in each community. The quality of an argument is not solely dependent upon its formal structure but more importantly on whether the argument is based on mathematical principles and evidence that are available to members of a community of practice and useful in extending the argument further. It does not mean, however, that if all students agree with an idea, it should be accepted as valid when it is not. In that case, the teacher's role should involve introducing new ideas to challenge students' conviction. In sum, a broader perspective on mathematical argumentation can be beneficial to understanding how mathematical knowledge could be developed and accepted in K-12 classrooms.

In contrast to numerous studies on secondary and post-secondary students' difficulties with proofs (Chazan, 1993; Harel & Sowder, 1998; Knuth, 2002), recent studies have shown that elementary students are capable of making fairly sophisticated mathematical arguments (Ball & Bass, 2000; Carpenter, Franke, & Levi, 2003; Maher & Martino, 1996; Stylianides, 2006; Strom & Lehrer, 1999). Two conditions should be noted in these studies. Support from teachers made mathematical argumentation possible even for elementary students. That is, the teacher's role was critical. Furthermore, argumentation activities in these studies were different from the way proof is taught and learned in high school. It involved social processes in which students talked to each other, tried to make sense of each other's ideas, and challenged each other's ideas by providing mathematical reasons. With adequate support from the teacher and through collaborative social processes, students in early grades were able to understand the limitations of using examples to make generalized statements and to work toward making deductive arguments.

Teaching Moves and Purposes

Classroom discourse is fundamental to teaching and learning. Classroom discourse practices are a patterned way of using knowledge of subject matter and tools of communication to develop knowledge in classrooms (Mercer, 1995; O'Connor, 1999). Accordingly, teachers' discourse moves can be considered "deliberate action[s] taken … to participate in or influence the discourse in the mathematics classroom" (Krussel, Edwards, & Springer, 2004, p. 309). Therefore, teachers' knowledge of content and purposes are expected to be expressed in teachers' discourse moves.

Recognizing the centrality of classroom discourse in teaching and learning mathematics, several researchers have used the lens of discourse to investigate classroom practices and teacher roles in traditional and reform-oriented classrooms. In a typical classroom in the U.S., classroom talk tends to follow a pattern of teacher initiation, student response, and teacher evaluation, often called IRE (Mehan, 1985). This pattern of classroom talk is especially well suited for directing students to particular answers and for checking whether students have acquired factual information. On the other hand, if a teacher wants students to engage in mathematical argumentation and become autonomous thinkers, her constant evaluation moves could interfere with helping students construct and critique their own arguments using their prior knowledge.

In contrast, in reform-oriented classrooms that emphasized communication and argumentation, alternative teacher discourse moves have been observed. Teaching moves that specifically support mathematical argumentation among students included establishing sociomathematical norms (Yackel, 1995, 2002), posing questions in a timely manner to press for justification (Martino & Maher, 1999; Stein et al., 1990), valuing disagreement (Lampert et al., 1996; Wood, 1999), and revoicing (O'Connor & Michaels 1996; Forman et al., 1998). In particular, revoicing is a useful teaching move that supports mathematical argumentation by aligning students with the content and each other so that students become positioned as constructor and critiquer in argumentation (O'Connor & Michaels, 1996).

The studies focusing on alternative teaching moves suggested that deep understanding of mathematics was required in executing these moves. However, their focus was not on investigating the knowledge that was required to make such moves or how the teacher used her knowledge in the course of these discourse moves. Instead, they focused on identifying the types of alternative discourse moves that were beneficial in supporting argumentation. Therefore, studies are needed that investigate how teachers' knowledge is used in teachers' discourse moves to support student participation in authentic mathematical discourse.

Mathematical Knowledge for Teaching

Teacher's mathematical knowledge is widely assumed to play an important role in teaching practices. Numerous studies have described the manifestation of teacher's content knowledge during classroom instruction (Cohen, 1990; Fennema et al., 1993; Heaton, 1992; Lampert, 2001; Swafford et al., 1997; Thompson & Thompson, 1994; Stein et al., 1990; Sowder et al., 1998). Teaching practices that a teacher's content knowledge mattered included pressing for explanation (Cohen, 1990; Sowder et al., 1998; Stein et al., 1990), posing open-ended questions (Swafford et al, 1997), focusing on conceptual understanding more than procedural understanding in teaching fractions (Sowder et al., 1998), responding to students (Hill et al., 2008), using mathematical terms and language (Stein et al, 1990; Thompson & Thompson, 1994), and the presence of mathematical errors in instructional explanations (Hill et al., 2008). While researchers believe that teacher's content knowledge has an impact on teaching practices, they have recognized that content knowledge by itself provided a limited view on the kinds of mathematical knowledge teachers needed to teach in ways that were productive for student learning.

Over the last few decades, researchers have made efforts to refine their understanding of the nature of teacher's mathematical knowledge. An approach to such efforts is a practice-oriented conception of teacher's mathematical knowledge, called Mathematical Knowledge for Teaching (MKT). It refers to mathematical knowledge that is pedagogically useful and unique to the teaching profession (Ball & Bass, 2003). For example, MKT includes knowing whether students' nonstandard strategies for solving multiplication problems are mathematically reasonable and generalizable. It also includes knowing how to use definitions in a way that is comprehensible to students and mathematically precise at the same time.

There is no conclusive evidence about a relationship between teacher knowledge and classroom practices and student learning. A recent large-scale study showed that MKT was correlated to student learning in the early grades (Hill, Rowan, & Ball, 2005). Hill and colleagues used a carefully designed multiple-choice assessment to approximate teacher knowledge and found teacher knowledge was in fact correlated to student learning. In contrast, other study has produced inconsistent results about the relationship between teacher's mathematical knowledge and student learning (Shechtman et al., 2010).

As we have only begun to understand teacher's mathematical knowledge that is useful for student learning and its impact on classroom practices, we still do not have a good grasp of how MKT comes into play during the course of teaching (Hill et al., 2008). A recent study investigated whether and how MKT is related to the quality of classroom instruction. Hill et al. (2008) found that teacher MKT was strongly correlated to the mathematical quality of instruction (MQI). Using in-depth case studies, they also found that other factors such as beliefs and the curriculum mediated the relationship between MKT and MQI.

While illuminating, MQI was designed to be generally applicable to the variety of instruction that can be found in mathematics classrooms. However, if a particular view on instruction is taken, then MQI may miss some of the nuances of context and purpose that would help us assess the effectiveness of a specific feature of instruction. For example, one dimension of MQI is the mathematical correctness of teachers' explanations. If purpose and context are taken into consideration, the explanations that are appropriate for their mathematical correctness may not have been appropriate. Say a teacher provides an explanation that is mathematically correct from an objective perspective. However, if it is not grounded on what has been previously established in a classroom, the explanation may be considered inappropriate as a legitimate argument in that specific classroom. Therefore, context is important in studying MKT because it allows us to carefully determine appropriateness of the use of MKT aimed at specific purposes teacher has in a classroom that shares history of building knowledge.

Along with ongoing efforts to identify how MKT is related to classroom practices and student learning in general, a recent study focused on specific practices such as mathematical argumentation. Yackel (2002) suggested that supporting argumentation in the classrooms from the elementary to the college level required that teachers have a deep understanding of mathematics and student cognition. This understanding included in particular the nature of mathematical justification, what constituted mathematical justification, whether one justification was mathematically different from another, and why something worked.

Stylianides and Ball (2008) also identified knowledge of situations for proving as a critical component of teachers' knowledge for supporting argumentation. It refers to being able to "identify situations in which proof is called for, recognize important mathematical differences among these situations, and stage appropriate opportunities for their students to engage in proving" (Stylianides & Ball, 2008, p. 311). As part of these recent attempts, more studies are needed in extending our understanding of what mathematical knowledge is needed for teachers to be able to support student engagement in authentic mathematical practices such as argumentation.

Statement of the Problem and Purpose of the Study

To date, research provides a limited understanding of what mathematical knowledge is needed for effective teaching in general (Hill et al., 2008) and for supporting mathematical argumentation in particular (Stylianides & Ball, 2008). Furthermore, while a broader perspective on mathematical argumentation for K-12 classrooms has been introduced and the value of mathematical argumentation in teaching and learning of K-12 mathematics has been emphasized (NCTM, 2000), there have been few studies that have focused on how classroom practices supported mathematical argumentation at the middle grades or lower levels.

This study was conducted as part of a larger research project, Bridging Professional Development, that focused on supporting classroom mathematical argumentation in middle grades in urban areas. Research questions for the larger project focused on whether professional development was effective in changing classroom practices. This study focused specifically on how participants' MKT came into play in implementing classroom mathematical argumentation given the assumption that MKT mattered in teaching practices. By adopting a multiple case study approach, the study was designed to explore teacher's MKT-in-action in support of classroom mathematical argumentation in middle grades.

RESEARCH QUESTIONS AND METHOD

The following questions guided the study:

1. What are the teaching moves in which teacher's MKT appears to be in action in support of classroom mathematical argumentation?
2. What mathematical knowledge for teaching is demonstrated in the teaching moves in support of student participation in mathematical argumentation?

These questions are addressed using data from multiple cases of 7th grade teachers who participated in a two-week summer workshop on mathematical argumentation with focus on proportionality and similarity. Because of the open nature of the research questions, a case study methodology was employed (Yin, 2003). Data included written assessments of MKT, semi-structured post-lesson interviews with video stimulated recall, and transcripts of classroom observation. They were collected both from the summer workshop and during and after teachers' implementation of the replacement unit on Similar Rectangle provided by the research team.

Because the focus was on classroom mathematical argumentation and teaching moves, this study used a fine-grained discourse analysis of classroom talk to understand the processes involved in constructing and critiquing mathematical arguments and the role of MKT in these processes. Episodes of mathematical argumentation in classroom discourse were located using a framework described in the following chapter and analyzed using an adaptation of Toulmin's scheme to fit the framework of classroom mathematical argumentation in this study (Toulmin, 1958/2003; Krummheuer, 1995). Teaching moves were identified and investigated to identify MKT-in-action in supporting student participation in mathematical argumentation.

SIGNIFICANCE OF THE STUDY

The study is important in three aspects. First, the study contributes to on-going efforts to understand the role of teachers' mathematical knowledge in teaching practices. The specific focus of the study on mathematical argumentation provides insights into the kinds of mathematical knowledge that teachers use to support student learning of mathematical argumentation in middle grades. Second, the study helps us broaden our perspective on mathematical argumentation and proof as a social process. While mathematical argumentation and proof usually has been studied as a psychological domain, this study approached mathematical argumentation and proof as a social domain by investigating the social, collective construction of argument by the teacher and students. Third, it provides insights into a possible framework for the trajectory of student development of mathematical argumentation in classrooms. The result is not meant to describe how mathematical argumentation should appear in middle grades in general though it may reveal some aspects. It only intends to provide insights into kinds of mathematical argumentation that could appear in middle grades and how it could be used in advancing and validating ideas by teacher with high MKT.

This study should be understood in context of urban education. The case teachers taught in urban schools that are characterized by low performance on standardized tests, racial diversity, low socioeconomic status, and high number of English language learners. Unless social or sociomathematical norms for argumentation were already in place, which was unlikely according to the research findings on classroom practices in urban schools (Mintrop, 2004), implementing authentic mathematical practices such as argumentation would be of difficulty for teachers. However, given the expectation that all students should have access to important mathematical practices, the study is important in providing insights into a possibility of implementing the practices for students who normally had less access to.

CHAPTER TWO: CONCEPTUAL FRAMEWORK

OVERVIEW

How teachers use mathematical knowledge in their teaching practices in general and practices of mathematical argumentation in particular is largely unknown (Hill et al., 2008). In this chapter, I start by providing an overview of two major perspectives on mathematical argumentation and proof, a traditional and an emergent one, drawn from professional mathematical communities and discussing their implications in K-12 classroom practices. Then I provide a conceptual framework for classroom mathematical argumentation based on the emergent perspective. In the second section, I discuss discursive approaches to the investigation of classroom practices drawn from sociocultural perspectives and what such research reveals about classroom discourse and teaching moves that support students' mathematical discourse such as argumentation. Finally, I discuss MKT by providing an overview of how conceptions of teacher knowledge have evolved in the last two decades and what research tells us about its connections to classroom practices and student learning.

MATHEMATICAL ARGUMENTATION AND PROOF

Mathematical argumentation is fundamental to the activity of doing mathematics. For this dissertation, I define mathematical argumentation as a process through which participants refine a claim through justifying activity until the mathematical validity of the claim is established. It is the core activity through which mathematical ideas have been advanced in the professional community (Hersh, 1997; Lehrer & Lesh, 2003; Schoenfeld, 1994). There are two main perspectives on what mathematical argumentation is.

The first, widely held perspective is that argumentation is the process of creating proofs. In this view, mathematical ideas are developed by formal logic (Hersh, 1997). The main activity in mathematics is believed to be deriving a conclusion from premises by logical deduction (NCTM, 2000). Therefore, a proof is made of a sequence of statements that are previously established as true such as axioms, definitions, or theorems, which are connected by formal logic, called deduction. No one would disagree that deduction is a key aspect of mathematics that distinguishes mathematics from other subjects. However, the view that mathematical argumentation is created by logical derivation has been criticized for its limitations in taking into account the intuition and creativity that mathematicians bring to the work of advancing mathematical ideas (Hersh, 1997; Schoenfeld, 1994).

The second perspective is that mathematical argumentation is a cyclic process of conjecturing, refuting, and refining conjectures through social interaction (Lakatos, 1976;

Thurston, 1998). In this view, mathematical ideas are advanced through heuristic methods and are subject to revision and refinement until their validity is no longer in question by participants. In his book *Proofs and Refutations* (1976), Lakatos illustrated how mathematical ideas were co-constructed heuristically through communication between a teacher and students. The heuristic process of co-construction involved participants engaging in the cyclic process of creating conjectures, challenging conjecture statements or assumptions for their validity using local or global counterexamples to further refine the conjecture, making explicit one's definitions and adjusting them to reject counterexamples, and adjusting properties to bear counterexamples as an exception to conjectures until statements were accepted among participants.

Polya's book *How to Solve it* (1945/2004) provides further insights into the role of heuristic methods in mathematical argumentation. According to him, heuristic methods are "not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem." (Polya, 2004, p. 113). He explained that heuristic methods are not like proof that verify truth of conjecture, but a method that people use during a discovery process to gain plausibility of a conjecture for its truth and to add certainty. Therefore, while heuristic methods do not ultimately verify whether a conjecture is true or false, let alone the philosophical question of whether ultimate verification of conjecture is possible or not, they play an important role in creating and advancing an argument and thus can provide insight for a deductive argument to follow. Heuristic methods include use of various examples, induction, and indirect proof.

Therefore, engaging in heuristic arguments can be a precursor to doing mathematical proof. For example, one might decide to start with a simple case to examine the plausibility of whether a statement is true or not and to gain insights into advancing an argument. This is not to say that use of a simple case can be accepted as a mathematically legitimate justification to verify a statement for its mathematical validity. Using simple cases, however, can be a useful way of starting an argument by explicating the mathematical structure underlying a problem that may be complex (Polya, 2004). Therefore, heuristic argument should be valued as part of argumentation activity for its function in providing insights into advancing arguments further.

Based on their classroom studies, Boero and his colleagues (1995, 1996) argued that conjecturing and proving are interrelated. In studies with middle grade students engaging in argumentation, they found students adapted the ideas that they worked with as they created conjectures and used them during constructing proofs. They hypothesized that there was “cognitive unity”, a close connection between reasoning underlying formation of conjecture and reasoning underlying production of proof.

On the other hand, Herbst’s study (2002) suggested proving and constructing proofs might be separate processes. He identified conflicting demands placed on the teacher when helping students prove a conjecture and produce two-column proofs in a high school geometry classroom. Not only did the teacher have to select a task to create opportunities for students to develop key ideas for proving, but he also had to help students to produce the two-column proof that the task required. The author suggested a lack of cognitive unity as a possible reason for which the teacher felt conflicts between supporting students to engage in proving and writing actual proof. Therefore, it is possible that creating proof may require a person to reorganize the ideas he or she developed through argumentation. While more studies are needed to understand how conjecturing, proving, and actual proof are related, the importance of the role of heuristic arguments as part of mathematical argumentation should not be ignored.

The second perspective views mathematical argumentation as a social process. Thurston (1998) argued that the validity of mathematical argumentation is determined by members in a community through their public negotiation, which means audiences and contexts matter. Hanna (1991) also stated that in a professional community there are no systematic criteria with which proofs are evaluated for logic. Instead, evaluation of one's argument was done through communication with other members of a community. Therefore, the validity of mathematical arguments may not be determined solely by formal logic or format but based on common grounds established through shared history of participants in developing knowledge in a community. Common grounds can be different from one community to another depending on the community's history, and is therefore subject to challenge. For example, the statement "three times eight is twenty four" may not be challenged for its validity in a 7th grade classroom but it will be challenged in a 2nd grade classroom. Also, a justification that may be accepted as legitimate in 2nd grade classroom may not be accepted as legitimate in 7th grade classroom because of differences in mathematical resources that are available in each classroom.

Therefore, the validity of arguments is determined by the shared history of members when developing knowledge in a community. If participants in the community can no longer contest it, the argumentation can be considered true and becomes publicly shared knowledge in a community (Hanna, 1991).

The importance of mathematical argumentation as an *integral* part of K-12 school mathematics has been raised in policy documents. The principles and standards (NCTM, 2000) recommends "reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied" (NCTM, 2000, p. 342). A recently published Common Core State Standards for Mathematics (2010) also recommends students develop proficiency in constructing arguments using previously established statements and critiquing others' arguments. While mathematical argumentation should be integrated into teaching and learning of any areas of mathematics, the standards pointed out that it should be taught with "different expectations of sophistication" across grades (NCTM, 2000, p. 56). For example, students in early grades may not be expected to construct the kinds of proof that appear in high school geometry textbooks. However, they can construct justification that may be less formal but mathematically reasonable to students. Furthermore, the standards recommend mathematical argumentation for K-12 grades entail student learning to "recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof" (NCTM, 2000, p. 56).

The National Research Council (2001) views mathematical argumentation as part of mathematical proficiency. Engaging in mathematical argumentation requires students to provide explanation and justification for their ideas. Mathematical learners should become engaged in activities in which they generate informed guesses, construct arguments to convince others, and defend their arguments by providing their own explanations and justifications. By trying to explain and exchange their ideas with others, students can make sense of their own thinking and that of others, which is important to the development of a productive disposition for mathematical proficiency (NRC, 2001).

Furthermore, the reason for the importance of mathematical argumentation for students is that students can be empowered by this process (Yackel & Cobb, 1995, 1996). Through the process of agreeing or disagreeing, students take charge of evaluating the reasonableness of explanations and justifications with support from a teacher. This means that mathematical validity is established by students who are engaged in the process of constructing arguments and, furthermore, that the students as a whole exercise authority regarding mathematical truth. This shift of authority is what makes engaging in mathematical argumentation empowering to students. However, this does not mean that a teacher should disengage from guiding students' argument and accept whatever they agree upon. Because students can have limitations in their reasoning as a whole, the teacher, as the ultimate judge, should step in whenever needed, not necessarily telling students what is right or wrong but posing questions to challenge their thinking.

A recent effort to conceptualize mathematical argumentation and proof for K-12 classroom has been underway. Based on two principles, intellectual honesty and continuum with the professional mathematics community, Stylianides (2007) developed a notion of proof for school mathematics that honors both the subject and students and is consistent throughout the grade levels. She defines proof as the following:

> Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
> 1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justifications;
> 2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291)

Her conception of proof for school mathematics is illuminating in conceptualizing mathematical argumentation for K-12 because of her awareness of social and contextual aspects of argumentation activity. However, it seems limited in conceptualizing argumentation from broader perspectives because it exclude the heuristic aspects of argumentation that can arise as ideas are created and refined. For example, she argued that an empirical argument should not qualify as a proof because it is not an acceptable form of proof in professional communities. By excluding heuristic aspects of argumentation, her notion is limited to proof as verification.

This limited view of mathematical argumentation has persisted in school curriculum and classroom practices despite the emergence of broader perspectives and attention to the importance of learning mathematical argumentation. Current curriculum and classroom practices seem far behind in acknowledging these perspectives of mathematical argumentation and integrating them as part of classroom mathematical practices in all grades. Not only does current mathematics curriculum focus mainly on a final form of argumentation—logical sides of mathematics—but proof has been taught as a topic of study in a restricted manner (Knuth, 2002; Lampert, 1990, 2001). Current classroom practices tend to focus exclusively on reading and writing proofs in a two-column format starting with previously written conjectures in geometry classes instead of proof as a tool for communication and creation of ideas in any area of mathematics.

Students' lack of experience with mathematical argumentation in early grades, researchers have suggested, seems to affect their experience with proof in later grades. Numerous studies have shown that students have limited views on what proof is and difficulty with doing proofs at the secondary and post secondary levels (Chazan, 1993; Harel & Sowder, 1998; Healy & Hoyles, 2000; Knuth, 2002). For example, in their study with high school students in UK, Healy & Hoyles (2000) found that students held two different conceptions on proofs. Students thought that algebraic proofs would receive the best mark. On the other hand, they preferred to use empirical or narrative arguments to construct arguments and for their explanatory power while recognizing the limitation of these empirical arguments for generalization. Healy and Hoyles suggested that students might gain conviction as they worked with empirical evidence while they might consider the format of arguments as an important aspect to consider in evaluating arguments.

On the other hand, some exploratory studies showed that elementary students are capable of making mathematical arguments (Ball & Bass, 2000; Carpenter, Franke, & Levi, 2003; Lampert, 1990; Maher & Martino, 1996; Strom & Lehrer, 1999; Stylianides, 2006). Early grade students' experience with mathematical argumentation and proof has been delayed till high school because young students have been assumed not to be ready for deductive arguments from developmental psychologists' perspectives. However, with adequate support from the teacher and through collaborative processes, even students in early grades are able to understand the limitations of using examples to generalize statements and to work toward making deductive arguments.

A Theoretical Model for Classroom Mathematical Argumentation

Here, I describe the theoretical model for classroom mathematical argumentation used in my study. Classroom mathematical argumentation is viewed as a social activity through which mathematical ideas are refined and validated. It is a cyclic process in which the teacher and students participate in creating a claim and refining it through justifying activity until participants reach an agreement about its mathematical validity, at which point the claim becomes shared knowledge or facts with mathematical legitimacy. Therefore, mathematical argumentation is a process of creating a mathematical argument, which in its basic form, consists of a claim and a justification as presented in Figure 2.1.

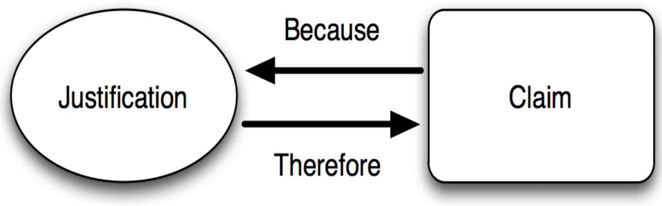


Figure 2.1. A basic form of argument

A claim is a statement that is subject to justification for its mathematical validity (Forman et al., 1998). For an argument to exist, there must be at least a claim statement and a justification statement. For example, a claim can be simply an answer to a problem, for which why the answer is correct is not known. A claim also can be a statement that is generated by observing patterns in numbers or objects, for which the reasonableness of the statement is in question.

To justify a claim means to "provide sufficient reasons for" it (NRC, 2001, p. 130). A justification can be in the form of a solution or statements that describe how a claim is generated and why it is true. Justification can be made using heuristic methods during the process of creating and refining the claim to add the plausibility of the claim. For example, students may start with a few examples to assess a claim for its plausibility. Students can use examples to find a pattern for generalization by gaining confidence about the validity of the claim. Justification can be questioned for its validity, which then becomes a claim to be verified.

A justification can be considered consisting of two elements: data and warrant (Toulmin, 1958/2003), presented in Figure 2.2. According to Toulmin (2003), *Data* or *Premises* are “the facts we appeal to as a foundation for the claim” (p. 90). Solutions to a problem can be considered data as it explains how the answer is arrived. *Warrants* legitimize the step between the data and the claim by appealing to a general rule or principle such as axioms or definitions. So, for example, warrant has to do with why a solution to a problem is a legitimate way to get to an answer. While recognizing that the distinction between data and warrants might not be always clear in real practice, Toulmin suggests that it can be drawn by looking at the difference in their nature. Data are explicitly requested when a claim is challenged whereas warrants are often implicit. Because data depend on warrants, warrants are used to produce an argument even though they are not necessarily present.

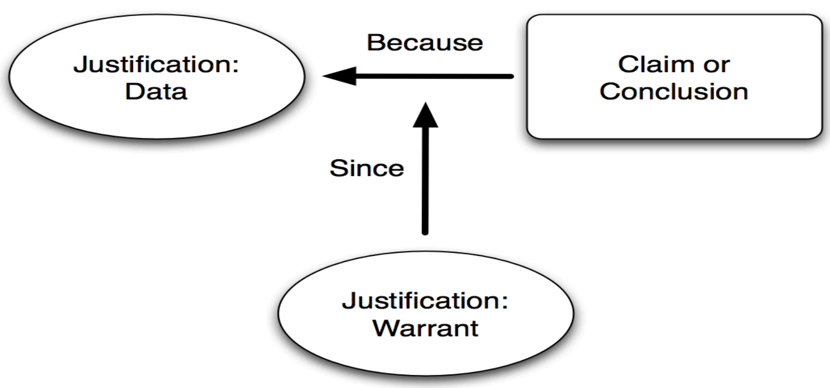


Figure 2.2. A basic form of mathematical argument showing two elements of justification

According to Harel & Sowder (1998), justification involves two sub-processes: ascertaining and persuading. If a student comes up with a claim but wants to know its validity, she will engage in justifying to ascertain the truth herself, therefore creating an argument. If there is more than one student involved and another student doubts the validity of a statement, participants will get engaged in justifying to persuade the contester. Therefore, the level of sufficiency of a justification depends on who is involved in establishing its validity (Thurston, 1998). For example, a justification that can be accepted in a second grade classroom may not necessarily be accepted in a ninth grade classroom.

In a classroom setting, if a statement is not acknowledged as a claim by other students or the teacher, we assume that it is personal knowledge left unexamined with respect to its reasonableness or validity (see Figure 2.3a). This can happen for a variety of reasons. It can happen when a teacher is simply unaware of the presence of a claim. Or it can happen when a teacher may be aware of its presence and decides not to pursue it because it is so basic so that there is no need to create an argument. For example, a middle school teacher might decide not to ask why the answer for 35 + 25 is 60 because the teacher assumes that most students know the answer and how to arrive at it. A teacher may also decide not to pursue an argument because it is not planned as a goal of the lesson or because she does not know what to do with it.

If students and the teacher agree with an argument without any challenge, then it can be taken as publicly shared (see Figure 2.3b). This can happen when participants do not see a reason to challenge because, for example, the argument, a claim and a justification, was so simple and clear that there is no doubt about its validity.

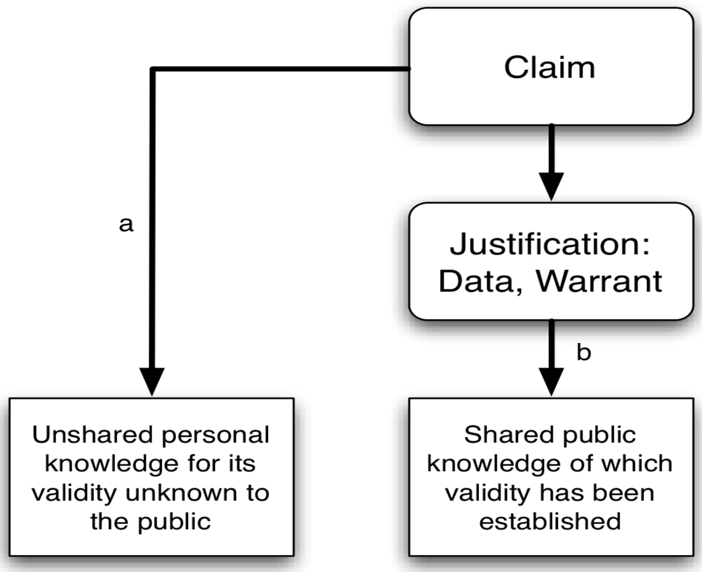


Figure 2.3. A basic process of argumentation

Argumentation can be expanded when a request for further clarification, an alternative justification, or a competing argument follows as presented in Figure 2.4. If someone, either a student or a teacher, asks the student who makes a claim to clarify or explain it further, students will have an opportunity to elaborate their argument by providing more information or a different justification, or by making explicit the implicit assumptions underlying the argument. Additionally, when an alternative justification is formulated, students can engage in a discussion in which they compare different kinds of justifications and reasoning. Having more than one idea and trying to make connections among ideas is a great opportunity for students to understand how others think and to extend their own ideas. The teacher's support is critical in helping students not only generate alternative explanations and justifications, but also understand mathematical similarities or differences among them (Yackel, 2002).

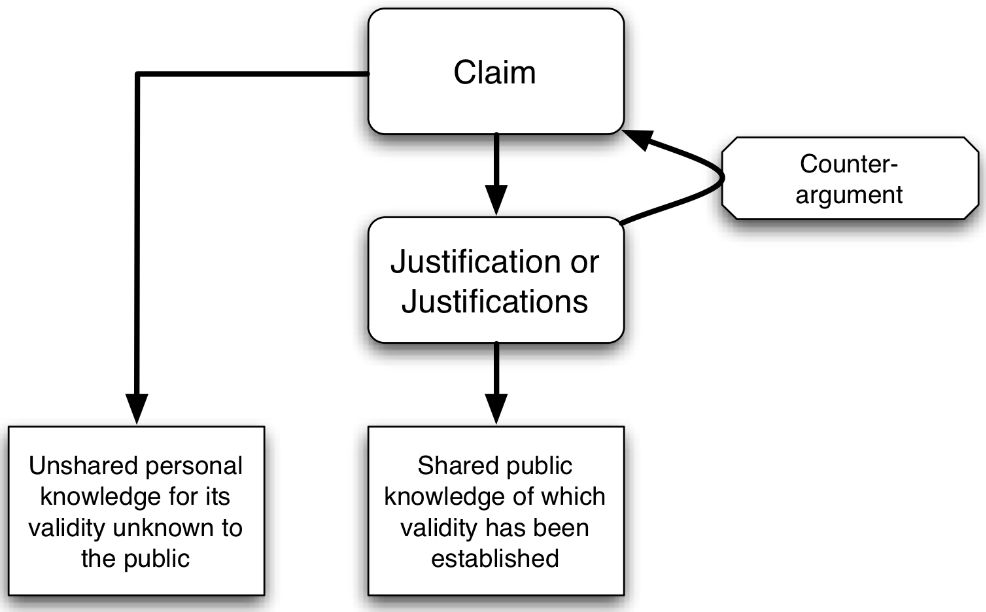


Figure 2.4. A theoretical model for classroom mathematical argumentation

A mathematical argument can be expanded also when someone disagrees with the original claim by suggesting a competing claim or counterexample, which can be considered a counterargument. Cognitive conflict is an important way to advance one's own understanding (Schwarz, Neuman, & Biezuner, 2000). As students try to resolve a conflict between the original argument and a competing argument, they are expected to reexamine their own ways of thinking, think of better ways to convince others, and try to come up with counterexamples to reject or stretch others' ideas. Students also may get engaged with examining the assumptions that they use in creating arguments. For example, students may need to make explicit a definition that they used to create their argument but left implicit. The definition then can become an object of public debate to accept or revise (Lakatos, 1976). Depending on the extent to which the original argument can stand up to refutation, the original argument will be refined to be made more plausible and convincing to participants. This means that students will need to go back to the original claim, revise the part that is invalidated by others, and repeat the process until there is no further challenge.

Argumentation is completed when there is no further challenge from participants questioning the validity of the claim. When participants reach an agreement, the claim becomes a theorem or fact, shared public knowledge of which validity has been established in a classroom. This does not mean that whenever consensus is reached, the argument is valid. Students might agree with each other's arguments for reasons that are not mathematical, such as deference to the social status of a particular participant (Civil & Planas, 2004). Students are also likely to be convinced on the basis of a few examples (Knuth, 2002). However, for mathematical validity to be established, argumentation should be based on mathematical reasons and be in a form of deductive argument in the end.

What are the signs of progress of argumentation? First, mathematical argumentation is taking place when non-mathematical statements including terms or justification are ruled out and imprecise statements become mathematically precise. Students may make an argument based on their everyday experiences and beliefs. Therefore, to make an argument more mathematically sound means to build on students' everyday language and to eliminate terms or arguments that are not mathematical. For example, to find similar figures, students can make a justification by saying that two figures are similar because they *look the same*. Students' everyday use of the term *similar* can come into play in justifying why two figures are similar mathematically. Also, students may provide justifications that are mathematically incomplete. For example, students might say, "I know it because I measured it." Eliminating such accounts as non-mathematical or imprecise justifications is a sign of argumentation. By constantly working to establish sociomathematical norms, the teacher can help students understand what qualifies as mathematical justification (Yackel & Cobb, 1996).

Second, mathematical argumentation is taking place when implicit premises or warrants that are important for justification become an object of public debate. Toulmin (1958/2003) assumes that a sound argument has to consist of three core elements: data, warrant, and claim. Data and warrant are the elements of justification that provides validity of a claim. Therefore, those elements should be made explicit and shared by participants to determine their validity. If students use warrants that are not previously established in a classroom or whose connection to the claim is less clear, then warrants can become an object of mathematical argumentation.

In dealing with children's arguments, however, not all information needs to be made explicit to advance their arguments. One study showed that children's arguments that occurred in natural settings were often missing some of the key elements (Anderson et al., 1997). They hypothesized that gaps in argumentation occurred because arguers did not provide information that might not be obvious to other participants. Furthermore, stating explicitly the warrant, "if p then q", that contributes nothing to the present argument, would be "superfluous". Therefore, those arguments should not be treated as having missing elements, and therefore, unsound. They concluded that children's arguments, presented in "q because p" or "p so q" should be considered sound by themselves because they implied the warrant, "if p then q" in their arguments.

Third, mathematical argumentation is taking place when it moves away from *authoritative* to *internally persuasive* argumentation. Referring to Bakhtin (1981), Wertsch (1991) explained two types of discourse that occur during interaction with another voice: authoritative and internally persuasive discourse. *Authoritative discourse* is fixed, one voice transmitting to another, without allowing new meanings to arise in response. On the other hand, *internally persuasive discourse* is open, inter-animated with other voices, and allows the creation of new meanings. Internally persuasive words are open to new interpretation and interact with other voices to create new meanings. Bakhtin's notion is helpful in understanding genuine classroom mathematical argumentation because the purpose of classroom mathematical argumentation is for students to gain a sense of plausibility and conviction about statements for their mathematical validity instead of teacher telling them what is right or wrong.

From a mathematical perspective, how can one distinguish internally persuasive mathematical argumentation from authoritative argumentation? Harel & Sowder (1998) identified three categories of schemes that students used in *ascertaining* oneself and *persuading* others: external conviction, empirical, and deductive. Students using an *external conviction proof scheme* depend on the authority of the teacher, textbook, or appearances of proof or symbolic manipulations with no relation to referents. Students with an *empirical proof scheme* depend on examples or perceptions including heuristic methods such as induction. Students with a *deductive proof scheme* make generalizations, use operational thoughts by forming subgoals and anticipating their outcomes, and use logical inference rules.

Students' use of external conviction scheme is evidence of engagement in authoritative argumentation because they simply accept as valid the justification that is presented by the teacher or textbook. Evidence for engagement in authoritative argumentation would be students accepting rule-based justifications without further requesting justification of why a rule is a proper warrant to the problem at hand. On the other hand, creating and accepting empirical or deductive proof schemes can be considered evidence of students' engaging in internally persuasive discourse. As numerous studies have shown, students preferred empirical schemes to gain conviction for themselves. Various heuristics of using empirical argument can allow students to give different interpretation of a claim in question and strengthen their conviction in the claim. Furthermore, dealing with deductive proof scheme can be considered evidence of students engaging in internally persuasive argumentation because deductive proof scheme involves using previously shared and mathematically validated statements, which requires other participants to agree with whether the use of such statements are legitimate.

TEACHING MOVES

Classroom discourse is central to developing knowledge in the classroom. It is a means by which teachers use their knowledge of subject matter and the tools of communication to guide students to talk and reason in the ways that the teacher intends for students to do (Mercer, 1995; O'Connor, 1999). A teaching move is defined as "a deliberate action taken by a teacher to participate in or influence the discourse in the mathematics classroom" (Krussel, Edwards, & Springer, 2004, p. 309). Therefore, teaching moves can be useful site to investigate teachers' use of subject matter knowledge in guiding students to become participants in mathematical argumentation.

Previous studies on classroom discourse have mainly focused on identifying patterns of interaction or teacher discourse in classrooms. Research has indicated that discourse in typical classrooms in the U.S. is characterized by the pattern in which a teacher initiates with a question, a student responds, and the teacher evaluates the response, often called IRE (Mehan, 1985). This pattern of classroom talk has some advantages for teachers: it helps the teacher to maintain authority by constantly evaluating their responses in relation to what she expects them to answer; and it makes it easy for the teacher to check on whether students have acquired factual information. However, with IRE exchange students rarely have opportunities to engage in building and defending their own ideas in classrooms (Hiebert & Stigler, 2000). The teacher's constant evaluation would interfere with helping students become autonomous in their own learning. On the contrary, less teacher interference gives students more responsibility for generating ideas on their own and evaluating each other's ideas. Instead of following up with evaluation, the teacher can use alternative teaching moves to open up opportunities for students to engage in creating, advancing, and evaluating their arguments.

Expanding on Bauersfeld's work (1980), Wood (1994) discussed two broad types of sequence of teaching moves in relation to the purpose they serve: funneling and focusing (Wood, 1994, 1998; Wood, Williams, & McNeal, 2006). In funneling, the teacher asks a series of "fill-in-the-blank" questions that lead students to the answer desired by the teacher. In contrast, in focusing, the teacher orients students to critical aspects of a problem while summarizing what students already know so that students can solve the problem on their own.

Using a sequence of moves, a teacher can accomplish her goal of supporting students to learn whether it is by leading students to the desired solution or by providing appropriate scaffolds to help students figure out a solution by themselves. In both types of moves, the teacher needs to use her own mathematical knowledge for teaching. To make funneling moves, the teacher needs to know ahead of time what a desired solution is. Also, to make focusing moves, the teacher needs to be able to notice critical aspects of problems to focus student thinking.

What is a main feature for mathematical argumentation that distinguishes it from other types of classroom discourse? The main feature of mathematical argumentation is the activity of justifying (Lehrer & Lesh, 2003). In classroom discourse, students can get engage in clarifying and explaining their ideas, which may not necessarily involve justifying why their statements should be mathematically valid. Mercer (1995) argued that justifying, which he described as a feature of exploratory talk, was the most productive type of talking in knowledge building in classrooms. He discussed three types of classroom discourse in knowledge construction: disputational, cumulative, and exploratory talk. In both disputational and cumulative talk, students engaged in co-constructing knowledge by challenging or elaborating ideas without talking about reasons. On the other hand, in exploratory talk, students engaged in making explicit reasons behind their ideas and making decisions about mathematical validity of their ideas, which he found the most effective in social processes of knowledge building. We know students engage in mathematical argumentation when they start doubting the mathematical truth of a statement and providing reasons.

What does research tell us about teaching moves that support mathematical argumentation? While there is increased recognition of the importance of the teacher's role in facilitating student participation in classroom discourse (Cazden, 2001; Cobb, Yackel, & Wood, 1992; Mercer, 1995; O'Connor, 1999; Stylianides, 2007), teaching moves that support student participation in authentic mathematical discourse such as argumentation has not been well characterized (Franke, Kazemi, & Battey, 2009). Some initial research has identified teachers' discursive moves that could be productive for mathematical argumentation. These moves included pressing for justification (Martino & Maher, 1999; Stein et al., 1990), valuing disagreement (Chazan & Ball, 1999; Lampert et al., 1996; Wood, 1999), and revoicing (O'Connor & Michaels, 1996; Forman et al., 1998).

First, posing questions to press for elaboration, explanation, and justification has been found to be an important teaching move that supports high-level cognitive activity such as argumentation (Kazemi & Stipek, 2001; Martino & Maher, 1999; Stein et al., 1990). When teachers probe students for their justification beyond producing correct answers, students' conceptual understanding is developed. Also, when teachers pose questions with careful sequencing by monitoring student thinking to probe for and challenge student justification, students can become engaged in reorganizing their thinking and revising their justifications, in which processes they advanced their arguments (Martino & Maher, 1999).

Second, by attending to the importance of conflict and resolution in knowledge growth from a Piagetian perspective, Wood (1999) identified teacher discourse practices that supported disagreement and resolution during classroom discussions in a second grade classroom. A teacher has to build expectations that students will become critical listeners able to respond to the ideas of others. Furthermore, the teacher has to make distinctions for students between social participation and mathematical participation in disagreeing with the ideas of others. Because disagreeing involves social interaction among students, students can use their everyday understanding of how to socially deal with conflicts. In the absence of teacher support, students may experience difficulties dealing with their peers to produce a mathematically appropriate resolution to a disagreement – that is, a resolution that is based on mathematical evidence and reasons instead of social status such as power (Civil & Planas, 2004; Lampert et al., 1996).

Chazan and Ball (1999) also discussed what was required for a teacher to manage disagreement among students. When students disagreed with incorrect reasons, the teacher had to step in and steer the focus of disagreement to important issues. She directed students to focus on definitions instead of how to calculate an average. Therefore, managing disagreement seems to require the teacher's understanding of students' conflicts in relation to mathematically important information such as definition.

Last, revoicing has been found to be a productive teaching move for students' mathematical argumentation. According to O'Connor and Michaels (1996), a teacher's conversational move "creates a participant framework," through which students are socialized into specific ways of talking and thinking by taking up the roles assigned to them by the framework. So they investigated a specific teacher's discourse move, revoicing, to see how the use of this move could evoke different participant frameworks from students. They found revoicing was useful for two purposes. The teacher used revoicing to reformulate student arguments in order to explicate student reasoning. Also, revoicing was used to create alignments or oppositions in an argument. By positioning, students took on roles of constructors and critiquers in argumentation (O'Connor & Michaels, 1996). By engaging in such conversations with the teacher, students were socialized into becoming active participants in mathematical talk.

Analysis of teaching moves can reveal teacher's knowledge of mathematics and instructional goals. For example, pressing students for justification requires the teacher's knowledge of what qualifies as mathematical justification. She needs to notice missing but critical elements in students' justifying statements to be able to ask for further elaboration. Also, analysis of teaching moves reveals the teacher's instructional and content goals. Building on O'Connor and Michael's framework on revoicing, Forman et al. (1998) illustrated how the teacher's instructional goal—being nondirective—and content goal—helping students understand generality of argument beyond applying algorithms—were manifested in the teacher's discourse moves such as revoicing. The findings implied that teacher's content goal required her own understanding of generality of arguments.

Therefore, teaching moves that support student participation in mathematical discourse can be a useful site to investigate teacher's mathematical knowledge in action.

MATHEMATICAL KNOWLEDGE FOR TEACHING

Teacher's mathematical knowledge has been widely assumed to play an important role in teaching practices and student learning. Over the last couple of decades, researchers have made efforts to conceptualize the teacher's mathematical knowledge that is useful in teaching practices. As an attempt to theorize teacher knowledge, Shulman (1986, 1987) introduced *pedagogical content knowledge* (PCK) as a knowledge domain that is specific to teaching. PCK refers to "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9). Teachers need to know not only the subject matter, but also how to form those domains of knowledge to make them understandable to students. PCK includes knowing which representations to use to introduce a concept, what difficulties students will have in understanding certain concepts, and how to make concepts relevant to students' diverse interests. Thus, knowledge of teaching mathematics entails having a fundamental understanding of mathematics and knowing how to use one's own understanding to meet students' cognitive and affective needs (Ma, 1999; Shulman, 1986, 1987).

Shulman (1986) also argued that as part of subject-matter knowledge, the teacher needed to know how ideas are developed and validated in the domain as well as facts, concepts, and principles. For example, besides knowing what a ratio is and how to solve problems involving equivalent ratios, the teacher needs to know where cross-multiplication came from, how it is related to the concept of equivalent ratios, and why it may or may not be accepted as a valid method in a classroom.

Using case studies, researchers have described areas of teaching practices where teacher's knowledge mattered. Such areas included the selection and use of language, elicitation of student thinking, and focus on concept versus procedure (Cohen, 1990; Fennema et al., 1993; Lampert, 2001; Swafford et al., 1997; Thompson & Thompson, 1994; Sowder et al., 1998; Stein et al., 1990).

First, the teachers' language use reflects the teachers' mathematical understanding. For example, Thompson and Thompson (1994) illustrated a teacher's struggle to express to students the meanings of concepts of rate and speed. The teacher's use of calculation oriented language made it difficult to facilitate students' understanding of the concepts. Furthermore, the teachers' use of inappropriate metaphors or analogies was another indication of limited understanding of the content (Heaton, 1992; Stein et al., 1990). She chose metaphors that were interesting to the students but did not accurately convey mathematical meanings.

Second, when teachers have knowledge of student cognition as well as content knowledge, they tend to be responsive to student ideas. An intensive investigation of a teacher revealed that when she taught a concept that she had weak knowledge of—fractions—she tended to ask directive questions and did not follow up with student comments. On the other hand, when she taught a topic for which she had participated in professional development focused on problem types and student cognition—addition and subtraction—she tended to be less directive and elicit more student thinking with a focus on problem solving strategies (Fennema et al., 1993; Lehrer & Franke, 1992).

Third, teachers with greater understanding of mathematics focused more on concepts than procedures in their instructional practices. A case study on middle grade teachers who participated in seminars that focused on developing proportional reasoning revealed that teachers began emphasizing more conceptual understanding than procedural understanding and asked more open-ended questions (Sowder et al., 1998). However, a teacher tended to slip back to teaching for procedures when she taught a topic for which she did not have strong content knowledge.

Building on Shuman's conceptions, researchers at the University of Michigan have attempted to conceptualize teacher's mathematical knowledge, called *mathematical knowledge for teaching* (MKT). It refers to "the particular form of mathematical knowledge that is useful for, and usable in, the work that teachers do as they teach mathematics to their students" (Hill et al., 2008, p.308). By analyzing teaching practices, Ball and her colleagues identified that teaching involved distinguishing between students' everyday language and mathematical terms, analyzing students' unfamiliar strategies or errors, explaining nonstandard procedures or mathematical rules, using effective representations, and choosing examples for effective learning (Ball & Bass, 2000, 2003). For example, a teacher needs to be able to distinguish between everyday use of the word *similar* and mathematical use of the term, which would help teacher anticipate pre-conceptions or confusion that students might express during instruction.

Findings from recent large-scale studies showed disparities in the relationship between teachers' MKT and teaching practices and student learning (Hill, Rowan, & Ball, 2005; Hill et al., 2008; Shechtman et al., 2010). Hill et al. (2008) found teacher's MKT was positively correlated to the quality of mathematics in instruction. The study focused on ten teachers using both quantitative and qualitative measures to analyze how their MKT was related to teaching practices. The researchers designed a framework for measuring the mathematical quality of instruction (MQI) by drawing on previous research on the teacher's content knowledge and classroom instruction. The elements of MQI included mathematical errors, responding to students inappropriately, connecting classroom practice to mathematics, richness of mathematics, responding to students appropriately, and mathematical language. They found that MQI was strongly correlated to teachers' MKT. Also, some mediating factors were identified such as curriculum materials and beliefs. In contrast to Hill et al.'s study (2008), Shechtman et al.'s study (2010) showed a lack of correlations between teachers' MKT and their teaching practices and student learning when using MKT assessments that were aligned for their mathematical content with teacher learning in PD and student curriculum. While more studies are needed to examine the assumption that the teacher's mathematical knowledge matters for classroom instruction, the mechanism of how teacher's MKT comes into play to support students' mathematical work in general or to support mathematical argumentation in particular needs further investigation (Ball, Lubienski, & Mewborn, 2001; Hill et al., 2008).

In sum, research provides evidence that classroom practices reflect the teacher's mathematical knowledge. Although the teachers' strong knowledge of mathematics in itself does not lead to high quality of instruction that supports student learning (e.g. expert blind spot), the teachers' content knowledge is generally believed to be a critical factor to ensure high quality of mathematical instruction for student learning.

MKT for Mathematical Argumentation

What has research told us about teachers' knowledge of mathematical argumentation and proof? Previous studies have focused on teachers' conceptions of proofs and were mostly conducted through interviews rather than investigation of teaching practices. The results of these studies were similar to results of studies on student conceptions of proofs. Knuth (2002) found that secondary teachers had limited conceptions of proof. For example, they failed to recognize the role of proof in promoting understanding beyond verification. Most teachers found example-based arguments more convincing than generalized arguments. They seemed to find a number of specific examples more reliable. Also, teachers depended on superficial features of arguments such as the inclusion of detailed information rather than mathematical substances to determine whether an argument was convincing. They viewed proof as a topic of study not as an essential tool of doing mathematics.

Furthermore, Ma's comparison study (1999) on practicing teachers in the U.S. and China indicated that the U.S. teachers were weak in understanding how truth is established and justified in mathematics. When dealing with unfamiliar problems, teachers did not seem to have knowledge of how to verify a claim mathematically beyond telling whether a claim was true or false. Instead, they tended to rely on the textbook to determine the validity without attempting to find out on their own. However, because these studies were mostly done using interviews without observing teacher's classroom practices, how their conceptions would play out in supporting mathematical argumentation and proof is still unknown.

By observing preservice teachers' discussions, Simon and Blume (1996) found that their limited understanding of multiplicative structures and mathematical justification appeared in how they interacted with each other and made sense of their peers' mathematical arguments. The fact that some preservice teachers were not persuaded by their peer's refutation by counterexample possibly indicated that they had limited understanding of the role of counterexample in proof. Teachers' limited understanding also appeared in their dependence on the idea that that the length times width formula for the area of a rectangle as a *fact* that could not be contestable for its truth. The study provided insight into what mathematical knowledge needs to be developed to support mathematical argumentation. The preservice teachers needed the opportunity to reexamine what they knew as "mathematical law" and learn to distinguish facts from claims, axioms from theorems and to have the opportunity to unpack their understanding of what is considered as fact and what can be contested for its mathematical validity.

What do we know so far about what mathematical knowledge is required for teachers to support mathematical argumentation? Yackel (2002) suggested that teachers need deeper understanding of mathematics. Specifically, her investigation of the teacher's role in establishing sociomathematical norms for argumentation from elementary to college level revealed that teachers need to understand what constituted mathematical justification, whether one justification was mathematically different from another, and why something worked. Furthermore, teachers needed to understand a conceptual trajectory of students' mathematical thinking. Besides understanding the mathematical quality of students' arguments, teachers needed to be able to identify what was currently available in student thinking on mathematical concepts in creating arguments and how to help them advance their arguments.

Researchers have just begun to identify by studying classroom practices what MKT would be important in supporting classroom mathematical argumentation. Stylianides and Ball (2008) identified knowledge that a teacher would need in order to create opportunities for students to engage in proving activity. In particular, a teacher needs *knowledge of situations for proving* as well as *knowledge of the logico-linguistic structure of proof*. Studies that have investigated students' and teachers' conceptions of proof have mostly examined knowledge of logico-linguistic structure of proof. While teachers need this knowledge in order to support mathematical argumentation, it is not specialized knowledge for teaching. On the other hand, knowledge of situations for proving is specific to the work of teaching. It refers to knowing how to "identify situations in which proof is called for, recognize important mathematical differences among these situations, and stage appropriate opportunities for their students to engage in proving" (Stylianides & Ball, 2008, p. 311).

Specifically, knowledge of situations for proving includes knowing different kinds of proving tasks and their relationship to proving activity. Understanding different kinds of proving tasks can involve knowing the number of cases involved in a task—single, finitely many, or infinite—and the purpose of a task—to verify or refute. Different proving tasks can in turn influence the kinds of proving activity that is initiated by the tasks. For example, a task that involves a single case to refute a claim can generate the activity of constructing a counterexample. Therefore, teachers need to understand both different tasks and how they influence proving activity, both of which constitute knowledge of situations for proving. Stylianides and Ball's study on the identification of MKT for proving activity is only a beginning to the conceptualization of the MKT that is needed specifically for classroom mathematical argumentation. As Stylianides and Ball noted, however, we are in great need of more studies that focus on what knowledge is needed for teachers to support proving activities and how it plays out in classroom instruction.

CHAPTER THREE: METHODOLOGY

DESIGN OF STUDY

The purpose of the study was to identify teachers' mathematical knowledge for teaching (MKT) that was needed students' mathematical argumentation in middle grades. Using an explorative case study with multiple cases, I specifically addressed the following questions:

1. What are the teaching moves in which teacher's MKT appears to be in action in support of classroom mathematical argumentation?
2. What mathematical knowledge for teaching is demonstrated in the teaching moves in support of student participation in mathematical argumentation?

My goal was to investigate teachers' discourse moves and to identify teachers' MKT in their moves that supported student participation in mathematical argumentation. Research suggests that teachers' profound understanding of mathematics is critical in supporting student engagement in mathematical argumentation, yet researchers have only begun to understand what mathematical knowledge matters in teaching in general and how it is enacted in classroom practices (Hill et al., 2008). Therefore, using a multiple case study design (Yin, 2003) with teaching move manifested as teacher utterance within argumentation episode as the unit of analysis, I identified and described teachers' MKT that appeared in their teaching moves to help students create and advance their mathematical arguments.

To ensure trustworthiness, multiple strategies were adopted (Lincoln & Guba, 1985). First, for transferability, purposeful sampling and multiple cases were used. Also, to increase credibility, triangulation of data from multiple sources and member checking were used. For triangulation, multiple data were collected, including MKT assessments, transcripts of post-lesson video-stimulated semi-structured interviews, and transcripts of classroom observations. Also, I debriefed my interpretations with other researchers for dependability.

Context

This study was part of a research project called Bridging Professional Development[2] that aimed at developing teachers' MKT and teaching practices that supported students' mathematical argumentation, with a content focus on ratio and proportionality. Mathematical conceptual framework for PD is provided in Table 3.1. The main purpose of the project was to investigate the effect of the professional development on participants' mathematical knowledge for teaching and classroom practices using randomized experimental design. Twenty-five middle school teachers were recruited for the project from schools located in an urban area of northern California. In June 2006, the teachers participated in a two-week summer workshop. In week 1, the participant

[2] The Bridging Professional Development is funded by grant number 0455868 from the National Science Foundation to SRI International (PI: Jennifer Knudsen, Co-PIs: Susan Empson & Nicole Shechtman).

teachers spent time deepening their mathematical knowledge on rate and proportionality. Then they were randomly assigned to two groups, either experimental or control group. During the second week, the teachers in the experimental group engaged in developing teaching practices that supported mathematical argumentation whereas the teachers in the control group engaged in studying how the concept of proportionality developed across grades. In the following school year 2006-2007, 15 teachers from both the experimental and control groups implemented a three-day replacement unit on Similar Rectangles. Teachers' implementation of the unit was videotaped by the research team.

Table 3.1

Mathematical Conceptual Framework for PD, Student Curriculum, and Assessment

- Distinguish between proportional and non-proportional situations such as distinguishing between multiplicative and additive structures in geometric figures.
- Know that two quantities are *proportional* when they vary in such a way that one quantity is a constant multiple of the other or when two quantities have a constant ratio. In similar figures, identify two constant ratios: scale and shape factors.
- Identify patterns among quantities in proportional situations such as $\frac{a}{b} = \frac{c}{d}$ = constant ratio, y=kx (a straight line through the origin), etc.
- Know that the linear function of the form y=kx (and the associated graph and table) is a proportional relationship between x and y.
- Know that the constant of proportionality is the constant ratio $\frac{a}{b} = \frac{c}{d}$ in proportion, the constant slope, k, in linear functions of the form y=kx, and the scale factor in similar figures.
- Make sense of *why* cross-multiplication works for proportional relationships.

As a research assistant, I worked on various aspects of the project. I assisted with developing a framework for argumentation, designing the summer workshop, creating MKT assessments and rubrics, and creating protocols for pre- and post-interviews. I interviewed teachers at the beginning and end of the unit, conducted a total of 36 classroom observations, and transcribed the interviews and observations for analysis. Finally, I assisted in data analysis of the interview transcripts, classroom observation transcripts, and MKT assessments.

Participants

For the current study, I selected a total of three teachers from the experimental and control groups. I purposefully selected the teachers on the basis of their scores on the MKT assessment for two reasons. First, I selected teachers ranked high in MKT because I wanted to investigate their mathematical knowledge in implementing challenging mathematical practices. Second, I wanted to lower the probability of incidents where teachers struggled with their own lack of knowledge, especially in handling simple mathematical errors that were not as critical as logical faults in argumentation. The description of three teacher's MKT from the assessment is provided in Table 3.2.

I included one possible contrasting case among the three by carefully examining their responses on one open-ended item on the MKT assessment. This item asked teachers to justify why rectangles built from repeatedly adding the length of the sides of original rectangle were similar (see Appendix A). All three teachers were capable of constructing generalized arguments in some way but their justification was different in nature. I noticed two different kinds of justification. One was a generalized justification that used variables to represent side length and cross-multiplication to evaluate the proportion. Another justification was not as mathematically complete as the first. However, teachers' written responses showed a justification with a few examples and noted on the patterns of repeated addition for generalization. I assumed that these two different responses might indicate something about each teacher's knowledge of mathematical justification and their own inclination on how to approach unfamiliar problems.

Table 3.2

Teacher's MKT Reflected in Open-Ended Items

Case Teacher	MKT scores on open-ended Items (Total of 100)	Mathematical Knowledge for Teaching
Nancy	49	• Be able to *partially* interpret key ideas such as within- and between-ratios in student response. • Be able to *partially* provide a correct mathematical definition of similar polygons. (e.g. "similar polygons have corresponding sides that are proportional.") • Be able to *partially* construct multiplicative relationships represented in different representation systems such as table, equation, and graph. • *Unable* to construct a generalized justification but able to justify with a few examples.* • *Unable* to justify a connection between linear functions and similar figures.

Table 3.2 (cont.)

Kelly	75	• Be able to justify a connection between linear functions and similar figures. • Be able to *partially* interpret key ideas such as within- and between-ratios in student response. • Be able to *partially* construct a generalized justification. • Be able to *partially* provide a correct mathematical definition of similar polygons. (e.g. "similar polygons are exactly the same shape. One of them is smaller than the other.") • Be able to *partially* construct multiplicative relationships represented in different representation systems such as table, equation, and graph.
Susie	83	• Be able to interpret key ideas such as within- and between-ratios in student response. • Be able to construct a generalized justification. • Be able to justify a connection between linear functions and similar figures. • Be able to *partially* provide a correct mathematical definition of similar polygons. (e.g. Similar polygons have the same angle but they have different size. Their ratios are proportional.) • Be able to *partially* construct multiplicative relationships represented in different representation systems such as table, equation, and graph.

Note. Nancy was able to construct a partially generalized justification in the post-test.

I ended up with two teachers from the experimental group and one from the control group. This selection was not intentional because the focus of this case study was not on the effect of the workshop between the groups. However, I anticipated that between the groups there might be differences in teachers' support of argumentation that would be reflected in their teaching practices.

I initially selected four case-study teachers, who scored above 70% correct on both closed and open-ended items in MKT assessment. However, after I began classroom observations, I decided to drop one teacher from my analysis. Because she spent significant portion of time managing the class instead of working with mathematical content, I realized I would not be able to get enough data to analyze mathematical discourse. Also, because of noise level of the class, it was hard to capture student utterances. Characteristics of the remaining participants and their schools are provided in Table 3.3.

Table 3.3

Characteristics of Case Study Participants and Their School

<table>
<tr><th colspan="3" rowspan="2">Teacher and School Information</th><th colspan="3">Teacher (Group)</th></tr>
<tr><th>Kelly (E)</th><th>Nancy (C)</th><th>Susie (E)</th></tr>
<tr><td rowspan="4"></td><td>Education</td><td>Highest Degree</td><td>Bachelor, Secondary</td><td>Bachelor, K8</td><td>Bachelor, Secondary</td></tr>
<tr><td rowspan="2">Teaching Experience</td><td>Years of Teaching</td><td>10</td><td>9</td><td>3</td></tr>
<tr><td>Grade Level</td><td>6-8</td><td>6-7</td><td>6-8</td></tr>
<tr><td colspan="2">MKT (group average with the total of 200)</td><td>165
(113.3)</td><td>138
(86.17)</td><td>153
(113.3)</td></tr>
<tr><td rowspan="2">School Information</td><td colspan="2">Free or Reduced Lunch (%)</td><td>30</td><td>26.6</td><td>76</td></tr>
<tr><td colspan="2">Statewide Rank for Performance (10 the highest)</td><td>6</td><td>7</td><td>4</td></tr>
</table>

Table 3.3 (cont.)

	Student population	ELL (%)		10	2.5	17.7
		Ethnicity	African American	28	43.5	40.2
			Asian	19	13.0	39.2
			Hispanic	16	10.8	14.4
			White	11	29.3	4.7

Note. Teacher names are pseudonym.

DATA COLLECTION PROCEDURES

During 2006-2007, the participant teachers spent 2-8 days implementing a replacement unit on Similar Rectangles, which consisted of three lessons: Copy Machine Enlargement to Similarity, Find Your Similar Rectangle, and Making Similar Rectangles, presented (see Appendix B). Mathematical goals for each activity in the curriculum are presented in Table 3.4. There were a total of 14 classroom observations made for all three teachers, with each observation taking 50 minutes. Three data sources provided information about teacher MKT and their classroom practices: (a) MKT assessment, (b) classroom observations, and (c) post-lesson interview.

Table 3.4

Mathematical Goals for Each Activity in the Curriculum

Activity	Mathematical Goals
Copy Machine Enlargement to Similarity	• Create a shared definition of similar figures, anchored in students' observations of images • Establish some methods for determining similarity by visual inspection
Finding Similar Rectangles	• Identify mathematical conjecture • Use multiple explanations of why two rectangles are similar
Making Similar Rectangles	• Use different tools to find different ways to explain or justify similarity • Generalize patterns

MKT assessment

MKT assessment was designed and validated by project personnel to measure teacher's mathematical knowledge for teaching on rate and proportionality. Out of 20 items, 10 items were multiple choice and another 10 items were open-ended. Responses for open-ended items were mainly taken account for the selection of case, which provided information about the nature of justification and definition. Examples items are provided in Appendix A.

Classroom Observation

Each lesson was videotaped using a stationary camera placed in the back corner of the room to capture both teacher and students as much as possible. While the focus of the study was teacher moves, the teacher moves were assumed to occur in response to student move. Therefore, the camera was placed at a location that could capture both teacher and students with the main focus on teacher. One wireless microphone was used to capture teacher utterances and three table microphones were spread out among student desks to capture student utterances. As a non-participant observer, I stayed in the back of the room controlling camera and volume when necessary and made field notes to capture contextual information that could be missed in videotaping but might be important for understanding episodes of classroom argumentation. When possible, I held a debriefing session with teacher for clarification of field notes.

Because the focus of the study was on teaching moves in interaction with student moves, teacher and students' verbal utterances as well as nonverbal aspects of the discourse such as writing and drawing were transcribed by verbatim for each observation. Also, contextual information about observation such as student involvement in whole class discussions and informal interviews with the teachers were added to each transcript. Also, since the study required a fine-grained analysis of classroom discourse, transcription was made carefully by two graduate students to precisely capture terms or language teachers and students used.

Post-lesson interview

The post-lesson interview was conducted for two purposes. One was to investigate teacher's thinking processes in making teaching moves, especially their intentions for specific moves. Another purpose was to triangulate the data from classroom observations, to make sure my interpretation was consistent with the teacher's thinking. The interview was conducted two to four weeks after the classroom observations. This interview was part of the bigger Bridging study, so the existing interview protocol was used. It was semi-structured and consisted of a pre-determined set of questions that addressed teaching moves, purposes of the moves, and what the results of the moves were. Interview questions are shown in Table 3.5.

Table 3.5

Post-lesson Interview Protocol

Purposes	Prompts
Overall Impression	• Can you talk about what teaching move or moves are happening here? Describe to me what you're seeing.
Probing for intentions and observations about outcomes.	• Looking at this in retrospect, what was your purpose or purposes in doing this? • What did it do?
Origin and context of the move.	• Do you use these types of moves often? • Can you tell me a little bit about that? (e.g., why or why not) • Where do you think you learned to make moves like that?

At times, teachers' answers were probed in depth to further elicit their thinking processes regarding their moves and mathematical knowledge. The interview was conducted using video clips that were selected by the observers including myself during transcribing and teachers responded to questions specifically with reference to the clip after they watched. Two to three clips were selected for each interview that included argumentation episodes and teaching moves that were either typical or unusual. The interview was audio recorded and transcribed.

DATA ANALYSIS PROCEDURES

As an exploratory case study, the goal of my data analysis was to identify teaching moves that supported argumentation and teachers' MKT in those moves. Themes were generated through a recursive process of analyzing the transcripts, checking them with other confirming or disconfirming evidence within and across cases, and modifying the initial themes (Marshall & Rossman, 2006; Strauss & Corbin, 2007). To increase credibility, triangulation of data from multiple sources and member checking were used. For triangulation, my interpretation with transcripts of classroom observations was cross-checked with other data such as MKT assessments and transcripts of post-lesson video-stimulated semi-structured interviews (Strauss & Corbin, 2007).

All classroom transcripts were coded to identify argumentation episodes and their mathematical content as part of the bigger study. Two coders were assigned to each transcript and discussed the findings for the purpose of inter-rater reliability. Then, I coded for teaching moves initially by using basic codes for argumentation such as request for claim and request for justification and generate other codes for teaching moves.

Through several iterations, I assigned codes for each teacher utterance. Then I investigated teachers' MKT demonstrated in the moves. My participation in analyzing several other teachers with various levels of MKT greatly informed the search for themes among the teachers in my own case study.

Coding Argument and Argumentation

An argument was located by identifying claims and justifications. A statement or a sequence of statements was coded as Claim when a justification followed. Figure 3.1 presents how argument was located. Once a justification was initiated in some way, it was coded as Justification, whether it was complete or not. The justification in combination with the claim were coded as an argument. For example, in Table 3.6, the statement in line 1 was not considered as a claim until the statement in line 3 (request for justification) or 4 (justification) occurred. If no justification followed, then a claim-like statement was noted as a candidate for a claim but not coded as Claim. Sample codes are provided in Table 3.6. As discussed in Chapter 2, no one would follow up with justification or request for justification if validity of a statement was already known to the public.

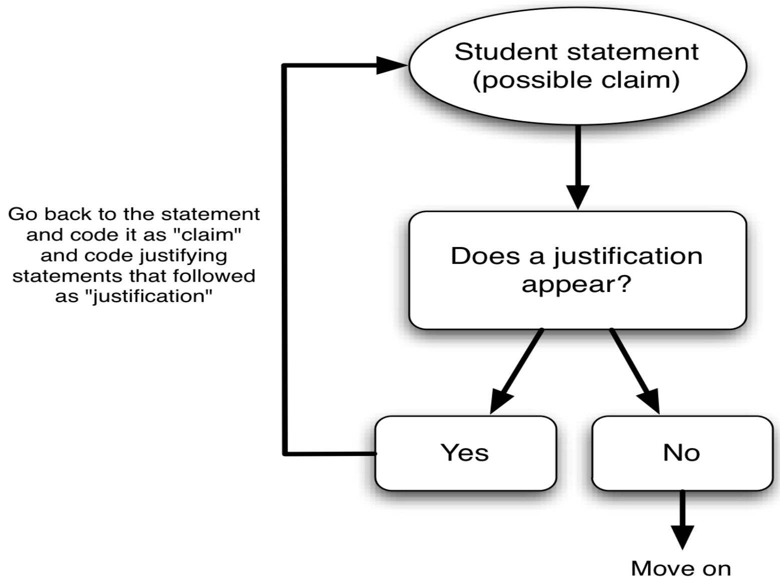


Figure 3.1. *Flowchart for locating argument*

In the process of identifying claim and justification, missing information was filled in if referents existed in the conversation or any non-verbal cues. In justifying statements, Warrant, if any, was identified using Toulmin's scheme, which was discussed in previous chapter. When a warrant was missing, previous discussion was searched for a possible warrant that students appeared to use implicitly in their arguments (Anderson et al., 1997). For example, in the example provided in Table 3.6, the warrant was considered implicit in the student's justification. The implicit warrant may go something like, if you had two rectangles where a ratio of a rectangle was made by dividing a ratio of another rectangle by a constant, they were similar. Even though the warrant was not made explicit in student justification, it was considered implicit because discussion of the warrant appeared in previous discussion as a definition of similar rectangles. The fact that students created a justification that was a direct application of a definition was considered as indicative of student use of the warrant.

Table 3.6

Sample Codes

	Line number	Dialogue	Elements of Argument	Teaching moves
Argument 1	1	T: There's another pair that is similar. Yes Kevin.	Argument 1 begins.	Request for a claim
	2	Kevin: C and D	Claim: "Rectangles C and D are similar."	
	3	T: C and D. and how do you know that?		Revoicing - acknowledgement Request for justification
	4	Kevin: Because C divided by 2 is <*inaudible*>	Justification	
	5	T: Okay, so C, six fifteen if you divide by 2, you get::		Revoicing – reformulation, Request for information in justifying
	6	Kevin: 3 and 7.5	Justification: "Dividing by 2 the ratio of C, 6 by 15, gives the ratio of D, 3 by 7.5."	
	7	T: You get 3 and 7.5. Okay.	Argument 1 ends.	Revoicing - acknowledgement
Argument 2		Any other pair?		Request for a claim

In each argument, justification was further analyzed for its mathematical nature using Harel & Sowder's proof scheme (1998) in combination with Balacheff's scheme (1988) on justification. If the justification appeared partial to assign a code for Justification, it was coded as incomplete. If the justification was incorrect, it was coded as incorrect. Otherwise, each justification was coded as (1) external, (2) empirical, or (3) deductive justification. Categories for Justification are presented in Table 3.7.

Table 3.7

Categories of Justification

Categories of Justification	Characteristics	Sub-categories
External justification	Students rely on an authority such as a teacher, textbook, appearance of the argument, or symbolic manipulations without meaningful referents.	• Authoritarian proof scheme • Ritual proof scheme • Non-referential symbolic proof scheme
Empirical justification	Students rely on perceptions.	• Use of perceptions
	Students rely on examples. (including heuristic methods)	• Inductive proof scheme • Simple or extreme example • Local counterexample
Deductive justification	Students rely on generality, operational thought, and logical inference. Accepted principles such as axioms are used to derive justification.	• Transformational proof scheme (including use of generic examples) • Counterexample • Axiomatic proof scheme

Note. This scheme is adapted from Harel & Sowder's proof scheme (1998, 2008), Balacheff's justification (1988), and Polya's heuristic methods (1945)

An episode of argumentation was determined by combining arguments that were connected. If there were arguments that were connected in such way that they supported or refuted the initial argument, those were included in an episode of argumentation. Once each episode of argumentation was identified, I chunked the episodes from the transcripts for further analysis for teaching moves and MKT.

While zooming in to locate episodes of argumentation, I also zoomed out to locate other mathematical discussion that did not qualify for mathematical argumentation in order to find out mathematical nature of the discussion. I found a few mathematical discussions that had the potential for mathematical argumentation but ultimately did not qualify. For example, I took out mathematical discussions that included mathematical claims for which no justification followed. Also, I noted episodes where students engaged in discussing definitions of similarity or similar rectangles, because of their potential use later as a warrant for argumentation.

Coding Teaching Moves & MKT

Once arguments were analyzed, I coded teaching moves considering their functions in support of argumentation and MKT that appeared in moves (Well, 1999). By following the work of conversational analysis (Heritage, 1984), I coded teaching moves by referring to the following turns provided by participants in conversation and evaluating functions of the moves. For teaching moves, I started with preliminary categories for argumentation such as request for claim, request for justification (Martino

& Maher, 1999; Stein et al., 1990), and request for consensus as well as revoicing (O'Connor & Michaels, 1996; Forman et al., 1998).

Then I used open coding processes to further identify teaching moves that were used in argumentation (Strauss & Corbin, 2007). A list of codes for teaching moves are presented in Table 3.8. Then, I used a recursive process consisting of detecting emergent themes for teaching moves and demonstration of MKT in the moves, checking them with other confirming or disconfirming evidence *within* and *across* cases, and modifying the initial themes (Marshall & Rossman, 2006; Strauss & Corbin, 2007). I triangulated the emergent themes with other data sources such as post-interviews and responses in MKT assessment to cross check the interpretation.

Table 3.8

Codes for Teaching Moves

Codes for Teaching Moves	Examples
Revoicing	• Rephrase student statement • Acknowledge or confirm student contribution simply by repeating • "So you are saying…" • "Carey says…"
Request for claim	• "which ones are perfect copies"" • "Which ones are similar?" • "How *would* we know if one is a blow up of the other?" • "What pattern did you find?"

Table 3.8 (cont.)

Request for justification	• "Why are they copy machine enlargement?" • "How do you know?" • "Give me your proof!" • "Can you explain…?"
Request for definition	• "Can you give me your definition of similar?" • "What is ratio?"
Request for information	• "What is the length?" • "Give me the ratio of A." • "Width of the <with lengthening of the sound "the">
Request for clarification	• "I'm not sure I follow." • "What do you mean…?"
Request for consensus	• "Do you all agree…?"
Request for counter-arguments	• "Anybody think that might not be right?" • "Anyone disagree?" • "Can you think of an example that won't be?"
Give suggestion	• "Measure the length and the width and see if they are proportional."
Give examples	• provide an example
Give explanation/Justification	• "Because sides are not proportional, they are not equal"

CHAPTER FOUR: FINDINGS

The following research questions were addressed in this chapter:

1. What are the teaching moves in which teacher's MKT appears to be in action in support of classroom mathematical argumentation?
2. What mathematical knowledge for teaching is demonstrated in the teaching moves in support of student participation in mathematical argumentation?

The discussion of the findings is organized into two parts: Argumentation and teaching moves. The first part provides an overview of argumentation episodes and the nature of justification to help discussion of teaching moves. The second part discusses in detail three teaching moves in which MKT appeared to be central in advancing mathematical argumentation.

OVERVIEW OF ARGUMENTATION

In this section, I provide an overview of two aspects of argumentation: the occurrence of argumentation and the nature of justification within classroom argumentation. This overview provides background information on how teachers used their teaching moves and MKT in support of classroom argumentation to develop a concept of similarity provided with a curriculum unit on Similar Rectangles.

Occurrence of Argument and Argumentation episode

The discussion of the occurrence of arguments and argumentation episodes provides a sense of when teachers saw opportunities for argumentation in developing the concept of similarity in relation to factors such as the amount of time spent on activities as part of the whole class discussion and curriculum.

First, the total number of arguments and argumentation episodes appeared differently for each of the case-study teachers. The difference seemed to be partly influenced by factors such as the amount of time spent on activities, the amount of time spent on discussion with the whole class, activities covered, and the length of the arguments. The frequency of argument and argumentation episode is presented in Table 4.1.

In Susie's classroom, arguments and argumentation episodes were observed more frequently than in the other two teachers' classrooms. First of all, she spent more time than other the teachers on whole class discussion, which might have contributed to the relatively high occurrence of arguments. Also, she covered all three activities with whole class discussion. She created more opportunities for argumentation for the latter two activities than the first. For the two latter activities, she had students present their arguments in pairs. After discussing the definition of similar rectangles, students in pairs reported their arguments using proceduralized definitions in Finding Your Similar Rectangle and Making Similar Rectangles. After students presented their arguments, the teacher simply checked the methods that they used. Incorrect arguments were rarely observed.

Table 4.1

Argument and Argumentation Episode

Teacher	Days spent on the unit	Total number of argumentation episode	Total number of arguments
Susie	4	19	39
Kelly	8*	13	19
Nancy	2	8	22
Total	14	40	80

* She spent 8 days on the whole unit. Only 5 observations out of 8 were selected for the analysis, which included considerable whole class discussion.

In the case of Kelly, despite numerous days spent on the unit, a smaller number of arguments and argumentation episodes appeared compared to other two teachers. One possible contributing factor is that she spent a significant amount of time on small group discussion, leaving smaller amount of time for whole class discussion. Argumentation episodes appeared evenly across the three activities. Most argumentation that occurred in her class tended to be quite lengthy, perhaps because students' responses tended to be short or incomplete and required numerous follow up questions to elicit a complete argument.

In the case of Nancy, a fairly high number of arguments and argumentation episodes appeared given the relatively short amount of time she spent on the unit. She spent most of Day 1 for whole class argumentation for the first activity whereas she spent most of Day 2 for small group discussion for the latter two activities. Also, in her class, a mixture of short and lengthy argumentation episodes was observed. The longest argumentation episode appeared when students attempted to build arguments about the quantitative relationship between enlarged figures.

The curriculum included tasks that prescribed students to provide justification. For example, a task contained within two activities, Finding Your Similar Rectangle and Making Similar Rectangles, asked specifically for two different justifications for why rectangles were similar (see Appendix B), so all three teachers had students engage in constructing two different justifications in the two activities. In Susie and Kelly's classrooms, arguments appeared frequently as a form of group presentations. On the other hand, in Nancy's classroom, most of the time was spent on small group discussion supporting arguments, but as a whole class she simply summarized findings to the students. Each episode appeared in these activities often consisted of two arguments, one claim with two justifications, as prescribed in the task. This pattern did not always occur in other activities.

Unlike other two activities, the Copy Machine Enlargement activity did not specify a prompt for justification. But in all three classrooms, teachers asked for justification when students gave a claim about a pair of money bills, which is illustrated in the following excerpt.

> T: What about the red money? That's the money from Nicaragua. So, yeah? The red money. Jason, what do you think about the red money? Is that a distortion or a blow up?
>
> Jason: Distortion.
>
> T: A distortion? But what makes you think it's a distortion?
>
> Jason: It looks kind of wide and stretched out.
>
> T: Which one looks wide and stretched out? The one on the bottom looks stretched out?
>
> Jason: Yeah.

After asking students to select whether a pair of money bills was a copy machine enlargement or not, the teachers always followed up by asking why they thought so. In all classrooms, arguments and argumentation episodes appeared in this activity in this manner. However, embedded arguments in argumentation episodes appeared in different manners from the ones that appeared in Finding Your Similar Rectangle or Making Similar Rectangles. Often, multiple arguments were present in an episode argumentation. For example, teachers invited multiple students to provide their justifications for why the two bills were a distortion or an enlargement. Sometimes, two opposing arguments appeared in the same episode. This was likely to occur when teachers asked for consensus or invited counter-arguments.

In contrast to those three activities, when students as a class engaged in creating a definition of similarity, arguments were rarely observed. Instead, teachers and students were mainly engaged in the discourse of clarifying meanings of and eliciting student understanding of the definition from textbooks.

In Nancy and Kelly's classrooms, however, teachers created an opportunity for argumentation when making transitions from Copy Machine Enlargement and definition of similarity. They provided an opportunity for students to construct an argument about a quantitative relationship by generalizing students' observation from enlarged figures as an attempt to build a class definition of similar rectangles from the context of enlargement. Teacher's decision of turning general discussion into argumentation is discussed in detail in the following section on teaching moves and MKT.

Nature of Justification within Argumentation Episodes

The discussion of the nature of justification provides insights into how teachers used argumentation in teaching and learning similarity. In most arguments, students provided justification upon a teacher's request for justification after students made claims. They provided justification primarily to explain why they found the two figures were similar. In almost all arguments, warrants were missing or were implicit in the student justification. When students used concepts that they discussed prior to their argumentation, warrants were considered implicit to their construction of argument even if they did not appear explicitly in their statements (Anderson et al., 1998; Erduran et al, 2004). The following excerpt shows a student argument with a missing but implicit warrant.

T: What about the red money? That's the money from Nicaragua. So, yeah? The red money. Jason, what do you think about the red money? Is that a distortion or a blow up?
Jason: Distortion.
T: A distortion? But what makes you think it's a distortion?
Jason: It looks kind of wide and stretched out.
T: Which one looks wide and stretched out? The one on the bottom looks stretched out?
Jason: Yeah.

In this excerpt, the implicit operating warrant could be "If something looks wide and stretched out in one figure, then the pair is not an enlargement". The structure of Jason's argument is shown in the following Figure 4.1.

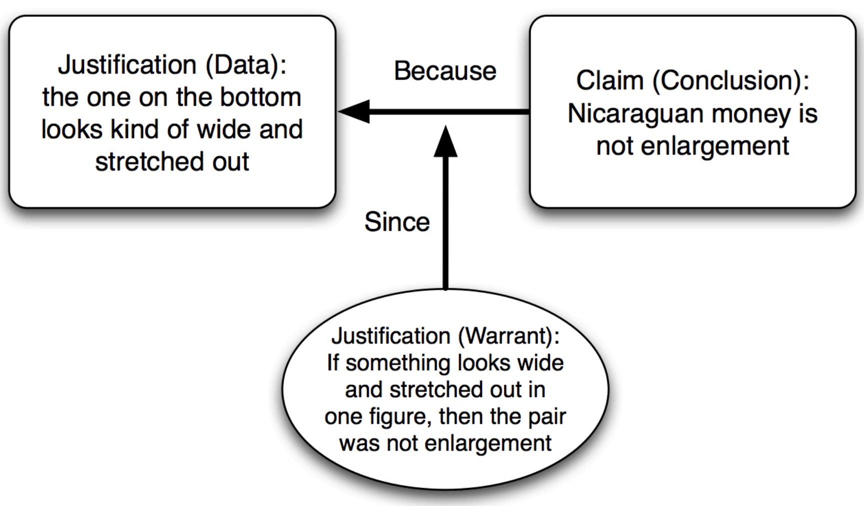


Figure 4.1. Structure of Jason's Argument

Overall, external and empirical justifications were used frequently across activities. Students tended to rely on external justification by using definitions from textbooks that were disconnected from the concept of Enlargement. For instance, students used cross-multiplication as a way to show that two rectangles were similar. In Nancy and Kelly's classrooms, students came up with the idea of using cross-multiplication as a way to show equivalence and the teachers accepted the justifications as valid. On the other hand, in Susie's classroom, the teacher suggested using the procedure. While cross-multiplication is a mathematically valid way of showing equivalent ratios and thus similarity, it does not embody the concept of enlargement or similarity. The process of cross-multiplying does not carry any meaning of co-variance. Students simply had to accept the procedure as valid to use in their justification without the chance to legitimize it through their own argumentation.

The nature of justification seemed to be partly affected by the content of the activities. For Copy Machine Enlargement, perceptually-based justifications appeared frequently and teachers accepted them as valid. As illustrated in the above excerpt, upon the teacher's request for justification, Jason provided a perceptual justification. Because the activity involved students' making sense of similarity arising from the real world context of enlargement, students were allowed to use their intuitive reasoning based on visual appearance of shapes.

The kinds of justification varied across classrooms. In Susie's class, in contrast to the other two classrooms, students tended to rely exclusively on external justification, using cross-multiplication or simplification of ratios. The following shows a common student justification for Finding Your Similar Rectangle.

> Peter: This big one [was] 8 inches and right here was 6 inches. The smaller one was 4 inches and by three inches. And then we did 4 over 3 times 8 over 6. Then we cross-multiplied. Then we got 24 over 24. And then the other way was, we did 4 over 3 and then 3 over 6 and then we divided 2 to each 8 and the 6, we got 4 over 3. Yeah we're done.

This kind of response was common because for this activity, Susie expected students to apply previously established definitions for similar rectangles in building their argument. Students needed to show that the two rectangles had equivalent ratios using two procedures, cross-multiplying or simplifying. In a sense, students were using previously shared statements as a class. On the other hand, they were simply accepting and applying proceduralized definitions that neither made sense in terms of enlargement nor were justified for their connection to enlargement.

In contrast to Susie, arguments observed in both Nancy and Kelly's classroom included a mixture of kinds of justification. Similar to students in Susie's classroom, the students used procedures such as cross-multiplication for their justification to show similar rectangles. On the other hand, they also used a shared definition that was constructed in a different manner from Susie's. Both teachers provided opportunities for students to build a mathematical relationship from the context of enlargement through argumentation where ideas of a constant of proportionality came up, which became a foundation for class definition for similar rectangles. Therefore, students' use of previously established statements such as applying a multiplying factor to both length and width in justification was considered deductive.

Also, in all classrooms, generalized arguments were rarely observed. Consistent with the findings of numerous studies on students' conception on proof (Healy & Hoyles, 2000; Knuth, 2002), students and teachers tended to rely on a few examples to justify a generalized statement. The following episode illustrates an empirical argument that students created with the support of the teacher.

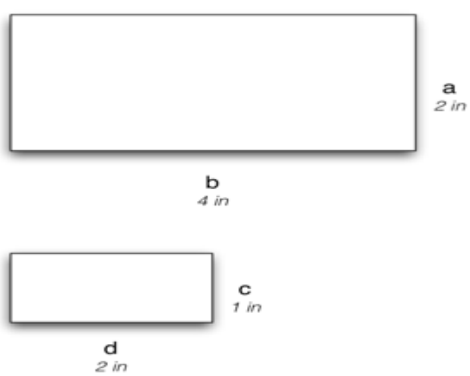
T: Do you think that all rectangles that have the length being twice as long as the width would be similar?

SS: No.

T: No? Can you think of an example, example that won't be. <*Nick's hand goes up*> Go ahead.

Nick: *c* could be three, *d* could be 4.

T: If *c* is 3 though, is this twice as big? <*indicating c and then d of a rectangle*>



$$\frac{a}{b} = \frac{c}{d}$$

Nick: No.

T: Okay so if *c* would be 3, then *d* would have to be <*pause, looking at SS*>.

S: 6.

T: 6. Yeah. So would that work? 3 over 6 <*writing* "$=\frac{3}{6}$" *next to the equation* $\frac{2}{4} = \frac{1}{2}$>.

15 S: Yes.
16 T: Are those are all equivalent <indicating $\frac{2}{4}=\frac{1}{2}=\frac{3}{6}$ >?
17 SS: Yeah.
18 T: Ah- okay. <*with facial expression of no surprise*>

In line 1, the teacher initiated argumentation to attempt to generalize that *all* rectangles that had a length being twice as long as the width would be similar, after showing the equivalence of the ratios of two rectangles, 1 by 2 and 2 by 4. Then she followed up with Nick, who opposed the claim by asking for a counterexample (Line 4). After helping him to realize his counterexample did not satisfy the condition that length was twice the width, she turned the dimensions of his example into ones that fit the condition. Then she asked whether all three rectangles had equivalent ratios, which could have been her attempt for generalization again (Line 16), but then she closed the argumentation by simply validating the claim using the three examples. In all classrooms, generalized arguments rarely appeared despite the presence of attempts from the teacher to elicit a generalization of a claim or a justification.

In sum, the occurrence and nature of arguments were affected by curriculum materials to great extent. Arguments tended to appear more frequently when the task specifically required students to provide justifications. Also, for the activity that required students to use their common senses, perceptually-based arguments tended to appear more often than in other activities of Finding Your Similar Rectangle or Making Similar Rectangles. Also, regardless of the curriculum material, arguments were observed with variances across classrooms. In one classroom, students relied heavily on external argument by accepting and applying definitions provided by teacher. In contrast, in the other two classrooms, students used a mixture of different types of justification. In the following section, teaching moves and MKT that were used in support of argumentation are discussed.

OVERVIEW OF TEACHING MOVES AND MKT

In this section, findings are organized into three teaching moves and their purposes where MKT comes into play in support of classroom mathematical argumentation: Revoicing by Reformulation, Responding to Difficulties, and Pressing for Generalization in Defining, as shown in Table 4.2. Also, the MKT that appears in each move is discussed.

Table 4.2

Teaching Moves, Their Purposes, and MKT

Teaching Moves		Purposes	MKT
Revoicing by Reformulation		Highlight core information	• Knowledge of core ideas about proportional relationships • Knowledge of student preconception
		Introduce new information	• Knowledge of how to validate ideas • (lack of) Knowledge of making connections between student statement and teacher reformulation
Responding to Difficulties	1. Decompose	Simplify	• Knowledge of a relationship between sub-argument and original argument
	2. Use counter-arguments	Challenge	• Knowledge of key ideas underlying student reasoning • (lack of) Knowledge of how to strengthen counter-arguments
Pressing for Generalization in Defining		Justify	• Knowledge of what can and needs to be justified • Knowledge of formulating mathematical definitions
		Provide	• Knowledge of student error in applying procedures • (lack of) Knowledge of formulating mathematical definitions

Revoicing by Reformulation

The first teaching move that supported mathematical argumentation was Revoicing by Reformulation. As discussed in the previous chapter, revoicing was defined as "a particular kind of reuttering (oral or written) of a students' contribution – by another participant in the discussion" (p.71) and reformulation was one of the functions of revoicing (O'Connor & Michaels, 1996). This move has been recognized as a useful teaching move in explicating student reasoning (O'Connor & Michaels, 1996; Forman et al., 1998).

The case-study teachers used revoicing by simply repeating what a student had said for the purpose of acknowledgement or confirmation. However, teachers also used revoicing to support student participation in argumentation. They used revoicing by reformulating student statements in order to highlight important information that was missing or implicit in student arguments. In other situations, teachers used revoicing by inserting information, which introduced new information that was completely missing in student statements. The latter was distinguished from the former in an important way. Insertion of information in her reformulation left conceptual leap between student statement and teacher reformulation.

The teacher's knowledge of teaching mathematics was reflected in their use of revoicing by reformulation to support mathematical argumentation in two main ways. First, the information that was made explicit through teacher's reformulation reflected teachers' understanding of what they saw as important in students' statements. To reformulate a student’s argument, teachers needed to use their own understanding of what was important but implicit or missing in the student’s argument. Second, a teacher’s insertion of new information in reformulating student statements revealed her knowledge of what qualified as valid mathematical justification. In the following sections I elaborate on these claims.

Revoicing: Reformulate to Highlight Core Ideas

One way that teachers used revoicing by reformulation was to highlight core information in student arguments. There were two different manners in which teachers used revoicing: reformulating by making implicit information explicit or reformulating by adding missing information without a conceptual leap.

The first episode illustrates how teachers used revoicing for reformulation to make explicit important information that was implicit in student arguments. This conversation occurred in the activity where the teacher, Susie, provided four rectangles and asked students to find a pair of similar rectangles.

T:	Okay. Volunteer. Who has a similar fig- rectangles. <*Sandy’s hand goes up*> Go ahead Sandy.
Sandy:	A, B, C.
T:	Okay just give me two pairs.
Sandy:	A, B.
T:	How do you know?
Sandy:	Because they both have <*inaudible*>.
T:	Okay, give me the ratios of A.
Sandy:	3 over 6 divided by 3 equals one half.

T: *<writing, "$\frac{3}{6}=\frac{1}{2}$" inside the rectangle A>* and this is:: *<pointing to rectangle B>*

Sandy: One half.

T: *<writing "$\frac{1}{2}$" inside the rectangle B>* Okay so they have the same ratios so they are similar then. Okay I'm sorry, that's similar *<erasing a congruent sign and writing "similar" between Rectangle A and B>* I'm just gonna write 'similar'.

After the teacher elicited a claim about which rectangles were similar, Sandy asserted that Rectangles, A, B, and C were similar. Without requesting a justification for Sandy's initial claim, the teacher asked Sandy to provide a claim and justification only on a pair of rectangles (Line 4). In order to show that all three rectangles were similar, students needed to start by constructing a justification for a pair and then constructing at least one more justification for another pair. Then they could either construct a justification for another pair or simply deduce a justification for another pair from a combination of pairs that were just proved, for instance, A~C from the fact that $(A\sim B) \wedge (B\sim C)$. However, the teacher did not pursue Sandy after a justification for one pair. She repeated the same process of initiating claim and justification on other pairs of rectangles with other students, without concluding all ABC were similar despite the later presence of independent arguments for AC and BC.

She restated Sandy's arguments in two ways, one in writing and another through speaking. After Sandy provided a reason, partly inaudible but possibly with missing details, she followed up with eliciting questions for details such as ratios of each rectangle (Lines 8 & 10). As Sandy spoke, the teacher wrote a reduced ratio of A and the ratio of B represented as a fraction inside the rectangles that were drawn on the board.

Then the teacher restated Sandy's justification as "so they have the same ratios" (Lines 13-14). Sandy used ratios represented in the form of fractions in her calculation to justify her claim. Also, she divided the length and the width of rectangle A by 3 and ended up getting the length and the width of rectangle B. As Sandy described her method of getting the same ratios, the teacher made explicit in her reformulation that the rectangles A and B had the same ratios, which was implicit in Sandy's statements.

Also, the teacher completed Sandy's argument by adding "so they are similar then" (Line 14). Considering the previous statement, the teacher's revoicing can be restated as "Because rectangles A and B have the same ratios, they are similar." The second part of her revoicing summarized the core of the justification and the claim together.

She repeated the same process of eliciting an initial claim and justification and following up with a restatement by filling in student arguments with information. By filling in, she drew students' attention to what was important in their arguments.

A second episode illustrates a teacher reformulating student arguments by adding information that was missing but which she deemed important to understanding the argument. This conversation took place during whole class discussion on justifying whether a pair of money bills was a copy machine enlargement or a Photo Shop distortion. The whole class discussion started with the teacher, Kelly, asking each small group of students to vote on whether the second bill in each pair was a copy machine enlargement or a Photo Shop distortion of the first bill. Then, pointing to a pair of Tanzanian bills, she invited a justification from the class by asking "How many people think that that one was on Photo Shop?" and "What is your justification?" Evan was one of the students who had attempted to provide justifications based on measurements as well as visual appearance, but was not able to complete his justification using his measurements. So the teacher turned to the whole class with the measurements Evan provided.

T: Okay I'll ask you question. I'm looking at the numbers Evan gave me.
S: Uh-oh.
T: Uh-oh, huh? Okay let me write them big. <*re-writing numbers bigger next to the table*> He had seven by two and a half for the little one, okay? And he had nine and a half by five for the big one <*writing, "7 x 2.5, and 9.5 x 5 below*> <*T looks at class*> and you know what I'm gonna say?
S: Oh!
<*A few hands shoot up.*>
T: What do I see? What do I see that makes me think maybe it's a copy machine?
S: Nine by five.
Evan: <*hands up*>
T: <*pointing Evan*> Yah? Yeah?
Evan: Because like um- <*pointing to the board*> because the five- you know, they both [are] halves, and then right, there is a half, and then there's two halves right there.
Tau: Oh! You know what I mean Ms T?
T: What are you trying to say? Tau, translate?
Tau: It's like 2.5 less on both.
T: Do you notice that? This one went up <*Evan: That's what I meant!*> by 2.5, and this went up by 2.5 <*writing, "2.5" below 9.5 x 5*>.
S: Yep. Yep.
T: Hmm, so is it a copy machine?
Tau: Yes it is.

25 T: Or photo shop?

As soon as the teacher drew the class's attention to Evan's measurements, students responded with "uh-oh," indicating that they noted something wrong. As indicated in lines 3-6, Kelly showed agreement with them, which confirmed for the students that there was something wrong with Evan's argument. Then she rebroadcasted Evan's measurements in oral and written form to the class. She did not add any significant information that led students to focus on the increase. She simply presented the measurements in a way that might have helped students see the constant increase in both length and width.

Then she invited students to justify the claim that the pair represented a copy machine enlargement (Lines 9-10), which was opposed to Evan's original claim. She first gave Evan a chance to provide his justification; he responded, "they both [have] halves" (Lines 14-16), which was a simple observation among the quantities but had nothing to do with showing whether the pair was copy machine. As Tau jumped in, the teacher did not follow up with Evan's argument, but redirected her attention to Tau (Line 17). After Tau stated in a short sentence that the pattern was that the measurements were decreased by 2.5, the teacher followed up Tau's statement with revoicing (Lines 20-21). This time, she rebroadcasted Tau's argument to the whole group adding some information. The added information involved referents for the quantities. The information that *each side* was increased by 2.5 was not as clear in Tau's statement as in teacher's reformulation.

She also changed “decrease”, which was implied from Tau’s use of the word “less”, to “increase” to fit the context for enlargement.

However, her reformulation of Tau’s argument with this added or changed information did not expand or shift Tau’s argument in any way. Her revoicing with the insertion of referents simply made Tau’s arguments more understandable to his classmates. Also, the change made by the teacher did not shift Tau’s argument in terms of reasoning. Then, again without evaluating the argument, the teacher posed a question about whether Tau’s argument justified the claim for a copy machine enlargement or not, a move to search for a warrant (Lines 23 & 25).

In both episodes, the teacher’s MKT was evident in what information was made explicit or added to student arguments through revoicing. In the first episode, the information that was made explicit was the warrant, “the rectangles had the same ratios” and the claim associated with this warrant, “they were similar”. This information, as the warrant, was a critical part of the definition students in this class shared. That is, to show that a pair of rectangles was similar was to show that they had equivalent ratios. Because the warrant in the student’s argument was implicit, the teacher made both the warrant and claim explicit to the class. The teachers’ revoicing move placed importance on the presence of warrant, in other words, use of shared definitions. Through revoicing, she accepted the student argument as legitimate.

In the second episode, the teacher added information that was missing or unclear in the student’s argument but which she deemed important for students to recognize.

When the student provided the terse observation that “It’s like 2.5 less on both”, the referent “both” were ambiguous. While it is likely that some students already noticed the same observation that Tau made, the teacher made it explicit to the class by adding the referents in her revoicing of Tau’s statement.

However, in both cases, the teacher’s reformulation of student arguments did not shift or expand the original argument in a significant manner. The teacher simply highlighted the core that was already built into the student arguments, but that was not made clear. The fact that teacher’s reformulation did not expand student argument in a significant manner is important in terms of who is in charge of the argument. Even with the teacher’s follow up moves with reformulation, the original arguments of the student remained as the object of argumentation. This aspect is also important to distinguish this first move from the second type. In the second type of revoicing, teachers introduced new information they wanted students to see or use with leaving a conceptual leap between student statement and teacher reformulation.

Revoicing: Reformulate to Introduce New Information

Another way of revoicing for reformulation was to reformulate student statements in order to introduce new information to students. When student statements lacked information, teachers tended to add the missing information in their revoicing moves. Unlike the first type, teachers reformulated student statements by adding information that was not apparent in student arguments. The added information was either something that teachers expected to see in student statements or something teachers needed to lead the discussion in a new direction they desired. However, there was a conceptual gap between student's original statement and teachers' reformulation by leaving out connections between the two. In the following discussion, I present two different manners of reformulating student statements by adding information that is missing in student statements, which resulted in introducing new information.

This first episode illustrates a teacher's reformulation of a student statement by adding information that was not clear in the original statement but which she expected to see included in justifying statements. This conversation occurred in the activity where the teacher, Susie, provided four rectangles and asked students to identify a pair of similar rectangles.

T: Okay can somebody give me another pair? <*SS hands go up*> Mike.
Mike: A and C.
T: How do you know that?
Mike: Because they are equal. They both are half.
T: 2 over 4, so you simplified and get one half <*writing* "$\frac{2}{4} = \frac{1}{2}$" *inside the rectangle C*>.

At the teacher's invitation, Mike presented a claim that rectangles A and C were similar (Line 2). Following the teacher's request, Mike provided a justification by saying "Because they are equal. They both are half." (Line 4) However, in his statement, some information was unclear. What did he refer to when he said "they"? How did he come up with "half" for both?

As indicated in lines 5-6, the teacher reformulated Mike's justification by filling in the information that was missing. She started with giving the ratio for rectangle C, which was 2 over 4. Then she revoiced Mike' justification by adding a statement, "you simplified", which he had not specified. The original ratio for the rectangle A was 3 by 6, but it was reduced to ½ in a previous discussion to show it was similar to rectangle B. So, it seemed that Mike saw that the ratio of rectangle C could be written as ½ and found it was the same as the ratio of the rectangle A. While Mike was able to find an equivalence of the ratios, it was not clear from his statement whether he actually simplified a ratio by dividing both numerator and denominator by 2 to show $\frac{2}{4} = \frac{1}{2}$.

Specifying the method, simplifying, which was missing in Mike's statement was critical to the teacher because the method had been discussed as part of the definition of similarity. So to show equivalent ratios or similarity, he should have used either cross-multiplication or simplification. But he did not make it clear how he had found the equivalent ratios in his justification. Therefore, regardless of how Mike had actually found the ratios to be equivalent, the teacher had to add this missing information that was important in validating his argument.

This exchange came after a discussion of the definitions of similarity. So a learning goal for the teacher might have been to help familiarize students with finding equivalent ratios using two procedures to find similar rectangles. Her choice of easy numbers probably helped students to find equivalence of ratios without difficulties. The only thing she needed to do in her follow-up for student justification was to check whether students found equivalent ratios and what procedures they used. If this information was missing or unclear, she had to add it in her reformulation of student justification to legitimize it.

The second episode illustrates how a teacher added information in her reformulation to lead students in the direction she intended. This conversation appeared at the last part of the whole class discussion from Susie's classroom on whether a pair of money figures was copy machine enlargement. Right before this conversation, a student made a claim that the pair was a copy machine enlargement but was not able to give a reason to support her claim. This conversation started with Kevin's opposition to her original claim.

Kevin: The smaller one is longer though. Right there it's shorter <*pointing to the larger Nicaraguan bill, vertically oriented* that one <*smaller one*> compared to the one <*bigger one*> <*inaudible*>.

T: Okay, everybody what I'd like you to do is- Katelin. Everybody eyes on me. Kevin is saying that the sizes are NOT proportional. <*pointing with hands both bills, with fingers widened*> So, what I'd like you to do is <*one hand up*> Come on. Excuse me. Using your ruler, measure the two-dollar bills. <*Nicaraguan money*> Measure the length <*pointing horizontal side*> and the width <*pointing vertical side*> and see if they are proportional.

As a response to Kevin's opposing argument, the teacher rebroadcasted Kevin's statements to the class by reformulating them. She did not repeat any of what Kevin said in her reformulation, but she inserted the technical term, “proportional”, to refer what Kevin meant. Her reformulation functioned as a move to lead students to quantitative arguments by expanding on Kevin’s descriptive arguments. After this conversation, she moved on to tasks involving measuring and setting up proportions. While it was not clear whether students also understood Kevin’s justification as an explanation of non-proportionality, her reformulation became an entry to lead students to the idea of using proportion to justify enlargement.

The purpose of her inserting the term can be interpreted in two ways. Her revoicing could be seen as a simple act of introducing a technical term to students. After all, it is the teacher’s role to help students understand and use mathematical language and terms with precision in describing mathematical phenomena (Hill et al., 2008). However, this does not seem to be the case in this instance. Her following moves indicated her intention to move on to calculate proportion. After inserting the term, she asked students to measure the lengths and the widths and to determine if they were proportional. In fact, the whole class discussion that followed had to do with setting up and calculating proportion. Therefore, her intention in inserting the term was more than a simple act of introducing a mathematical term.

Rather, it is highly likely that she wanted to move students to go beyond perceptually based evidence to a mathematical way of determining if the pair of bills represented a copy machine enlargement. Her intention to lead students to the idea of using proportion was confirmed with her post-observation interview. She wanted to lead students to think about using proportion. So her revoicing move became an initiation into moving from perceptually-based arguments into mathematically advanced ones. After she rebroadcasted her interpretation of Kevin's argument to the class, she set for the full group the task of measuring the side lengths of the small and the large objects and setting up proportion.

Two issues arose. Kevin's *voice*[3] was not heard in the public space as the conversation between Kevin and the teacher appeared to be private. It is only the teacher's voice that resonated in the public realm, even if she gave credit to Kevin by saying "Kevin is saying". It is hard to say that Kevin's voice was delivered to other students in her reformulation because of missing connection between Kevin's argument and her reformulation.

This missing connection has to do with "horizontal mathematization", a process of transforming a real world phenomenon into a mathematical one (Rasmussen et al., 2005). Kevin's argument appeared between two different kinds of arguments, perceptually based arguments and mathematically based arguments. Before Kevin's argument, students provided arguments based on visual observation. Then Kevin started a quantity-based argument but in a descriptive manner. The teacher's revoicing moves after Kevin spoke, however, left a conceptual gap between Kevin and what followed. How Kevin's argument and teacher's reformation were equivalent was not discussed at all. Therefore, her insertion of the term was only used to accomplish her goal to make

[3] The term, "voice", is written in italics to indicate its meaning from Wertsch (1991) as a form of thinking and speaking that represents a perspective.

transition to quantitative arguments, which left a conceptual gap between Kevin and the follow-up arguments, or between qualitative and quantitative argument.

In both episodes the teacher's knowledge appeared in what she accepted as valid justification. In the first episode, she added an explanation of method that she expected to see in student justifications but was missing. This missing information was part of the shared definitions of similarity in this classroom, albeit a proceduralized one. Therefore, the teacher made sure through her reformulation of student statements that in order for a justification to be validated, the shared method should have been used regardless of whether the student actually used or not. Also in the second episode, the teacher added information to lead students to bring up a specific kind of justification. However, she did it in a way that left a conceptual gap between the student's original statement and the teacher's reformulation.

Instead of paying attention to what students were thinking, the teacher seemed to take an authoritative stance by highlighting core elements that were not in the student argument but that she expected to see. However, the connections between the student argument and her reformulation that were left out may have been a indication of her limited knowledge of the connections between student cognition and mathematical concepts. She failed to create an opportunity for students to advance their arguments mathematically by building on their primitive arguments. Her limited knowledge in making connections was apparent in later discussion on the concept of ratios. When a student asked what ratios were, she was able to give a mathematically precise definition of ratio as “comparison of two quantities”. However, she did not discuss how the definition of ratios was conceptually connected to their discussions with enlarged figures.

In sum, there were two different ways of using the move revoicing for reformulation to support argumentation: highlighting core information or introducing new information. In both types of reformulating, the teacher’s understanding of the core of student arguments appeared. Also, added information in the teacher’s reformulation revealed her knowledge of what counted as valid arguments. However, the teacher’s limited knowledge also was revealed in the missing conceptual connection between student statements and the teacher’s reformulation of student statements. The findings are summarized in Table 4.3. In the following section, I discuss the second move that MKT appears to be in action to help students advance their arguments.

Table 4.3

Revoicing By Reformulation

Teaching Moves	Purposes	MKT demonstrated in teaching moves	MKT
Revoicing by Reformulation	Highlight core information	Key ideas for proportional reasoning such as "having equivalent ratios", within- or between-ratios Key ideas underlying student conceptions such as "adding the same amount to both sides" for additive reasoning	• Knowledge of key ideas in proportional reasoning • Knowledge for validation
	Introduce new information	Previously shared ideas such as definitions or procedures	• Knowledge of how ideas are validated • (lack of) Knowledge of making connections between student statement and teacher reformulation

Responding to Difficulties

Another teaching move that was used to support classroom mathematical argumentation was Responding to Difficulties. When students had difficulties in constructing or refuting one's arguments, teachers responded to the difficulties in two ways: decompose arguments to simplify arguments or use counter-arguments to challenge.

The first move is similar to one of the heuristic methods of problem solving, *decomposition* (Polya, 1945). If a problem has a complex structure, one strategy to solve it is to decompose it into smaller pieces that are manageable. Likewise, when students had difficulty in constructing arguments, teachers decomposed the original question by breaking it down so students could start with a sub-argument. By decomposing, teachers reduced the cognitive complexity of the original question for argument.

Another move is to use counter-arguments to challenge students' current arguments. One way to use counter-arguments was to invite opposing arguments to create cognitive conflicts between opposing arguments. Also, teachers provided extreme cases to exaggerate the critical part of arguments to help students focus on the feature teachers wanted their students to see. This move is also related to a heuristic method that Polya (1945) discussed, *specialization*. By examining special cases, students could gain insight into the original problem.

In creating sub-arguments, one teacher used her knowledge of the definition of similar figures, which appeared in her response to item 4 in MKT assessments (see Appendix A). The teacher seemed to use her knowledge of the definition in breaking the original argument in that the two sub-arguments reflected the two parts of the definition of similarity, size argument and shape argument. Also, her limited understanding was observed in terms of justification using generic cases and precise meaning of the definition.

In providing extreme arguments, teacher's MKT was evident in their choice of cases that could exaggerate the core feature of main argument, so that students focused on an important feature such as the relationship among quantities. Teachers' knowledge of the concept of proportionality was used in creating extreme arguments that were parallel to the main arguments in terms of structure. However, one teacher's limited understanding was observed in her failure to bring out critical ideas in proportionality such as ratios about why the extreme argument disproved subtraction claims.

Responding to Difficulties: Decompose into Sub-Arguments to Simplify

One way teachers responded to student difficulties was by breaking the original argument into sub-arguments. Instead of letting students work directly with the main argument, teachers decomposed the original one so that students could construct sub-arguments.

The following episode illustrated how a teacher broke down the main question when students had difficulties in constructing arguments to the question. This discussion was part of students presenting their arguments about whether there was an enlargement for each pair of money figures. The earlier discussion showed signs of student misunderstanding of "perfect copies" and "machine-made copies", the terms the teacher, Susie, used to describe copy machine enlargement. After some clarification, she posed a question to the whole class about a pair of images of Nicaraguan money as to whether they were perfect copies.

T: Okay shh… let’s look at those two dollar bill- what is that <*Nicaraguan money*>? Ten dollar bills? Those are- what are they? Are they perfect copies or are they NOT a copy?

S: They are not.

T: Why do you say they are not- why do you say they are not perfect copies?

S: Blurry.

T: They are blurry. Forget about the blurriness. We are talking about the length. And the width. Focus on the length and the width. Okay so these two, are they similar? <*pointing to Nicaragua money*> Or are they perfect copies? Are they congruent? Think about it that way.

SS: No.

SS: Yes.

T: Kate, are these two congruent?

Kate: Where?

T: Ten dollar.

Kate: Compared to this? <*pointing to smaller Nicaraguan money*>

T: No these two <*pointing to bigger and smaller Nicaraguan moneys*>, just two. Those pair.

S: The little one and the big one.

Kate: Yeah.

S: No.

Kate: What?

S: This and compare to that <*talking to Kate*>

Kate: Oh. No.

T: They are not congruent. Why are they not congruent?

S: Because one’s bigger than the other.

T: One’s bigger than the other. Could this <*smaller Nicaraguan*> be a shrunken version of top bill <*bigger Nicaraguan*>?

Kate: Yeah.

T: Yeah. Okay, why is that?

Kate: <*Kevin’s hand up*> Because either one is shrunk or enlarged.

With the teacher’s request for justification, a student said they were not perfect copies because they looked “blurry” (Lines 4 & 6). In the earlier discussion, there was another student who presented the same argument. It is possible that because of the terms "perfect copies" or "machine made copies" the teacher used to describe the task of finding copy machine enlargement, students were confused. They might have thought from everyday experiences that they were supposed to find whether a pair of figures generated through a machine could make perfect copies or not, instead of thinking about enlargement. In both incidences where students presented arguments based on this kind of confusion, the teacher did not follow up with any move that gave or requested an explanation of why the argument with blurriness was not acceptable. Instead, she ignored the arguments and redirected students' attention to lengths and shapes.

As students' confusion and difficulty persisted, the teacher broke down the question from whether a pair of figures was similar to whether a pair was congruent (Lines 7-10). A study showed that when students gave incorrect responses, a teacher tended to ask questions that required less cognitive complexity in student responses (Nathan & Kim, 2009). Therefore, pedagogically speaking, it is possible that the teacher decided to break down the question of similarity into the question of congruence so that students could construct arguments for a simplified question. As a way of dealing with student difficulties, she broke down the main question into a question with less cognitive complexity so that students could make an argument with less difficulty.

The teacher's move of breaking an argument into two sub-arguments might not be the sole pedagogical decision to make it easy for students to respond. The way she broke down the original into two separate arguments required her use of MKT. There were three possible ways to think about her use of MKT in this decision. One possibility is that she might have thought that she could disprove enlargement by disproving congruence, which is false logically while its inverse is true. This is a common logical error that people make in logic. However, this did not seem to be the case because the following conversation did not indicate her attempting to draw any argument to invalidate similarity from the argument of incongruence. After Kate's argument about congruence, the teacher followed up with questions of whether one figure was a "shrunken version" of the other (Lines 27-28).

Another way to think about her decision to break down the original argument from a mathematical perspective is her use of the idea of working with a generic case, one of justification strategies (Simon & Blume, 1996). Because congruent figures are a generic case of a family of similar figures, she might have intended to help students start from an argument with congruence and build from there. This possibility seemed unlikely. She did not press for an argument about keeping shapes in building up from congruence arguments. Also, in a later activity where students were expected to find another pair that was similar to their rectangle, the teacher told students not to find a congruent figure. She said to students who found a congruent pair that, "They are congruent. [They] Gotta *[sic]* be different size. Remember similar rectangles have different size." Her explanation indicated that she did not consider congruence a special case of similarity, which was consistent with her partial understanding of the definition of similar figures shown in MKT assessments.

Instead, it is likely that she decomposed the original question into sub-arguments to control the conditions in the original question. To find whether one figure was an enlargement of the other, two conditions should be met: (1) same shape and (2) same or different sizes. Instead of having students to deal with two conditions at the same time, by breaking down the original question, the teacher helped students think of one condition at a time. While Kate still seemed confused with the task and the concept of enlargement, other students started giving arguments that involved ideas of proportionality.

Polya (1945) discussed decomposition as a heuristic method for Problem Solving. According to him, if a problem to solve has a complex structure, one strategy is to decompose an original problem into smaller pieces that are manageable. Likewise, when students had difficulties in constructing arguments, teachers decomposed the original question by breaking it down so students could start with a sub-argument. By decomposing, teachers reduced cognitive complexity of the original question for argument.

Therefore, her move of breaking original argument to sub-arguments was to reduce cognitive complexity of the original argument using a heuristic method of decomposition. It was unlikely that she founded her decision on either logical fault or knowledge of justification by using a generic case. By breaking the argument into two sub-arguments, she seemed to intend to help students focus on one feature at a time concerning shape and size, knowing that there were two features that were involved in concept of similarity.

This move seemed to be based on her MKT of two aspects. One has to do with her knowledge of heuristics methods. She used decomposition to deal with student difficulties in constructing arguments for enlargement. By decomposing, she helped students to work with sub-arguments by dealing with one condition at a time that underlies the concept of enlargement at a time. Also, her decision for breaking the original argument into two sub-arguments was based on her knowledge of the definition of similarity. In a later discussion with definition, the class talked about a definition of similarity as having the same shape but not necessarily the same size. Therefore, in showing similarity or enlargement, she used the definition to help students build two sub-arguments, size argument and shape argument, by controlling conditions underlying the concept of similarity.

On the other hand, her limited MKT was observed in her failure to address how sub-arguments were related to the original question. Even though students were able to provide an argument for congruence, why the question of congruence was related back to the original question of enlargement might not have been clear to students' mind. As discussed, she possibly used the definition of similarity in her mind to bring up sub-arguments. However, she did not make clear how congruence argument was related to the main question.

Responding to Difficulties: Use Counter-Arguments to Challenge

Another way of responding to difficulties was to use counter-arguments to advance student thinking. There were two different ways to use counter-arguments. One was to use counter-arguments that were invited from students. By inviting counter-arguments, teachers created opportunities for students to deal with cognitive conflicts among opposing arguments. Also, teachers used counter-arguments by giving extreme examples to advance students' current understanding.

The following episode illustrates a teacher's response to difficulties by inviting counter-arguments from students. This conversation occurred after students had discussed whether a pair of money bills represented an enlargement or a distortion. In a previous discussion, students' justifications were based mostly on visual appearance. So to press students to move beyond perceptually based justifications, the teacher, Nancy, invited David to share his idea after posing a question to the whole class "Do you have any idea how we'd know *for sure*?" She knew from the group discussion that David had created an argument based on a quantitative relationship between the measurements. After establishing the measurements of the bills, she posed a question that initiated conjecturing a relationship among the measurements.

T: Well so, if that's the case, how would we know if these two <*pointing measurements wrote on the board*> are exactly, if one is just a blow up of the other? If this is a blow up of this one? How would we know that? <*6 sec pause*> <*A few go hands up*> Carey?

<On board>

Lg	T-T	8.3 cm	Sm	T-T	4.9 cm
	Knees	4.1 cm			2.5 cm

Carey: You subtract the smaller measurements from the larger measurements and see if it's the same amount.

T: Okay, Carey says if we subtract the same thing from this one <*T points at the "lg" measurements on the left*>, we should get the same things here <*T indicates the "sm" measurements on the right*>. So subtracting the same thing from both measurements. Joey?

Joey: I was going to say that when you subtract the 4, the two numbers, whatever the difference is, whichever number is closer to the difference would be the original.

T: I'm not sure I follow. What would you do exactly?

Joey: Like if you subtract 2.5 from 4.9 you get 2.4. And then 2.5 would be closer to the 2.4. So whichever <*is closer?*> to 2.4.

T: So you're subtracting these two. Are you all okay with this? So if we subtract this from this <*indicates on the left "leg" side subtracting the 4.1 from 8.3*>, and this from this <*indicates on the right "sm" side subtracting the 2.5 from 4.9*>? We should get the same amounts?

When the teacher elicited claims and counter-arguments about relationships between measurements among enlarged figures, she followed up with revoicing by aligning students with content and with each other, which is another function of revoicing (O'Connor & Michaels, 1996). After Carey presented her claim, the teacher followed up with revoicing of her claim by starting with "Carey says" (Line 8). This way, she aligned the student with the subtraction claim. The teacher rebroadcasted Carey's claim to the whole class so other students could position themselves for or against Carey's claim.

Her revoicing of Carey’s statement made three matters explicit. The first is that by using “if … then”, she reorganized Carey’s statement to make the premise and conclusion clear. The second is that by pointing to the referents in figures as she revoiced, she clarified the quantities that were involved with the operation of subtraction. Finally, she repeated a part of Carey’s claim in a shortened sentence, which she saw as the core of Carey’s argument, “subtracting the same amount”.

After collecting two claims based on additive reasoning, the teacher continued the whole class discussion by inviting counter-arguments to challenge the subtraction claims.

T: Are you all okay with this? Anybody think that might not be right? Harry?
Harry: It won’t be the same though.
T: It won’t be the same. Okay.
Harry: Just because subtracting the length of it doesn’t mean <*inaudible*>.
T: So you’re saying that subtracting you don’t think will work?
Harry: No.
T: No? Why not?
Harry: I don’t know. I don’t know, I lost my idea.
T: Okay. If, if. Leslie, go ahead.
Leslie: It wouldn’t work.
T: Why not?
Leslie: Because if you have a larger amount.
T: I’m sorry, Leslie. Can I ask this table to please be quiet? Because I can’t hear what she’s saying. I don’t know how everyone else can. Go ahead.
Leslie: Like the larger it is, they’re larger amounts. The difference is larger. If they’re small, and the difference is going to be smaller if they’re really <*inaudible*>.
T: So it might work?
Leslie: It might. But like.
T: Is there something you can think of that might work? And actually, could you repeat what you said one more time. Because I want everyone to hear this. That was really great thinking.

Leslie: Okay, so. For the larger one. The measurements are larger so the difference is larger. And for the smaller the measurements so the difference is smaller.

T: Look, class. She's saying, you've got this big gecko, right? She's saying the difference between the length and this distance. Let's call it, almost the width. Right? The knee cap width? Should be bigger than the distance from here to here because the image is smaller.

David: But the differences...the kneecap width measurement should have gone up by the same amount that the tongue to tail-tip width measurement did.

T: Mm hmm. So how would we know if they still are, if this is still going to be a blow up? How would we know that?

At the teacher's invitation for counter-arguments, two students presented their opposition. First, Harry joined the argumentation by expressing disagreement to the part of the subtraction claim. By saying "it won't be the same though" (Line 23), he apparently disagreed with the part that "we would get the same amounts". The teacher used revoicing with Harry, who presented counter-argument. She revoiced Harry's argument by positioning him – this time against the original argument. Positioning students with or against an argument is one of the ways revoicing is used other than revoicing by reformulation discussed in the previous section (O'Connor & Michaels, 1996). The teacher first made clear Harry's position against the subtraction argument by revoicing "subtracting you don't think will work?" (Line 26) and followed up by asking for his reasons," but he was not able to give a complete arguments. Through revoicing, she juxtaposed the subtraction claim and Harry's claim by positioning Harry against the original claim.

Then Leslie joined the argumentation by expressing her position that the subtraction would not work (Line 31). With the teacher's prompt for justification, she provided her argument that lent her understanding of proportionality though it was seemingly partial. Leslie seemed to understand larger measurements should get bigger by larger amount than smaller measurements if the figures were to be proportional. However, she did not advance her argumentation further by not including how much proportion should be increased. Because of some missing information in the transcript, it was not clear what the teacher tried to accomplish when she asked that the subtraction might work. It is likely that she wanted to make sure that subtraction would not work in any circumstances from Leslie's argument. Because in previous discussion, she had never focused on what was involved in subtraction but subtraction itself. Also in later discussion appeared her emphasis on two operations, multiplication and division, that created similar rectangles.

The teacher then rebroadcasted Leslie's argument to the class, hoping that students would be challenged with their conviction with subtraction. David did not seem to be challenged with Leslie's counter-argument and put forth his argument with more detailed explanation. The teacher was frustrated that the other students did not seem to be challenged by Leslie's argument but rather convinced still with the subtraction argument. So after this episode, she provided other argument herself to challenge the students.

Her follow up moves using revoicing by positioning as well as reformulating after student statements shows a pattern of "focusing" (Wood, 1994). She did not ask a leading question to direct students in a certain direction to invalidate subtraction arguments. Instead, she simply repeated the core of the original argument to highlight the focus for public debate. She also made clear student positions, either for or against subtraction. Using revoicing, she helped students to focus on the core of arguments without leading them.

The teacher's MKT was demonstrated in her repetition of the core of opposing arguments. She clarified the core of each student's argument to make it the focus for the whole class argumentation. She wanted students to recognize that "subtracting the same amount" could not be a case of enlargement and needed to be invalidated by rebroadcasting Leslie's argument for disproportionality.

She did not seem concerned about which relationship students were referring to in their subtraction arguments, either between-relationship or within-relationship as she did not differentiate between Carey and Joey's arguments. Because she only pointed to each referent without explicitly stating it when revoicing the arguments, she did not seem to consider between- or within-relationship between the referents as critical as "subtract the same amount," possibly because it did not affect the core of the additive argument.

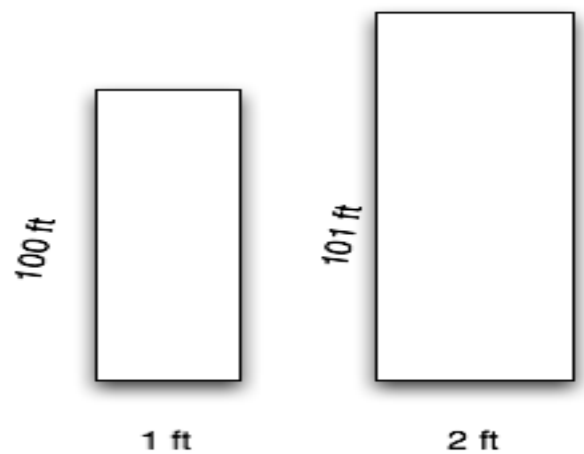
Dealing with counter-arguments seems to require teacher's MKT that is beyond knowing underlying reasoning or misconception of each argument. Counter-arguments are a great way to refine and advance initial claim (Lakatos, 1976). The presence of opposing arguments can create a space that cause cognitive conflicts to arise among students, which then can expand and transform current understanding (Schwarz, Neuman,

& Biezuner, 2000; Wood, 1994). However, the display of opposing arguments did not seem to create disequilibrium in student thinking. In dealing with opposing arguments the teachers' knowledge of, or lack of, using counter-argument to challenge students' current understanding was evident. Despite their knowledge of noticing the core of incorrect and counter-arguments, teachers failed to challenge student conviction with the additive claims. The teacher's dealing with opposing arguments seemed to require the teacher's knowledge of how to utilize and strengthen counter-arguments to handle the students' resistance of changing their current understanding.

In addition to using counter-argument by inviting counter-arguments from students, another way of using counter-arguments was by presenting extreme cases. Teachers used this move to help students recognize important features to advance their thinking. Because some of the arguments arising from the activities were hard to work with because they included messy quantities, students might have had difficulties in focusing on behaviors and patterns among quantities involved. Therefore, teachers provided extreme examples to exaggerate the critical feature of arguments to help students recognize the feature. By observing special cases, students could gain insights into the original problem.

The following episode illustrates a way in which a teacher challenged students when their difficulties persisted in figuring out whether the subtraction argument was valid. This particular conversation occurred right after the class discussed for quite some time two opposing arguments, arguments using subtraction strategies and the opposing arguments in determining an enlargement. As David who was one of the arguers who put forth the subtraction argument insisted, the teacher, Nancy, initiated conversation to the class by bringing up an example that was parallel to the subtraction arguments.

T: Well, you know what? Let me give you a simpler example. If you have something like this, two rectangles that are similar, okay? And I'm going to use a very extreme example. <*T draws two rectangles on the board.*> Let's say I make this 100 feet, and this is 1 feet. And this is absolutely not drawn to scale. And let's say here <*on the right side, she erases one of the rectangles she has drawn and redraws a larger one*>. 99 feet. No, sorry, you know what? No, I don't want to do that. I want to make this 101 feet, and I want to make this 2 feet.



T: I've increased this one <*length*> by 1 foot, and this one <*width*> by 1 foot. Would this one <*rectangle on the right*> be a blow up of this one <*rectangle on the left*>?

Carey: Yes.

T: Yeah?

Carey: Yes it would be.

T: It would be. This one <*referring to the width of the rectangles*> has gotten twice as big. This one <*referring to the length*> has gotten one foot bigger. These are not drawn to scale at all. But this one has gotten one bigger, this one has gotten twice as big. So I don't know if we can say these two are, if this is a blow up of this one. It would be a distortion. I'm just asking you to think about the whole thing where you're adding and

> subtracting amounts. Okay? I mean I think you're doing some great thinking. Let's think about that a little bit more, alright?

Instead of facilitating the discussion by inviting both arguments and probing, which she did in the previous discussion, she challenged students by bringing up another argument. Because students were not able to invalidate the subtraction arguments by themselves, she played the role of a critiquer and challenged the students' conviction with the subtraction arguments.

The importance of teacher's role as a facilitator is often emphasized in implementing reformed practices where student autonomy in constructing and evaluating their own argument is valued (Forman et al., 1998; Walshaw & Anthony, 2008). However, teachers tended to often misinterpret that reformed practices required them not to tell anything at all to give autonomy to students (Chazan & Ball, 1999). Instead, the teacher's active role is assumed especially when students have difficulties in figuring out by themselves whether ideas were valid or not. As student difficulties arose in invalidating incorrect arguments by themselves, teachers participated in argumentation not by directly correcting student arguments but by presenting related arguments to challenge the student's current understanding. The following post-interview reflected why she decided to change her role from a facilitator to a critiquer.

> So basically I was incredibly frustrated at this point. Because I think before this, there was a kid in the back, David, who is very, very bright. And he had this idea. And everyone in the class seemed to grab on and run with it…The subtracting one really was making sense to them.

Even though there were some students who disagreed with the subtraction arguments, the teacher found Leslie's counter-argument was not convincing enough to disturb other students' conviction with the subtraction argument. Therefore, she turned herself into an arguer to challenge the subtraction argument.

She did so by presenting an extreme case that could exaggerate disproportional increase due to subtraction of same amounts. She presented two rectangles with measurements of 1 by 100 and 2 by 101. She specified both the length and the width were increased by one, which matched "subtracting the same amount", the underlying structure of the subtraction arguments. Then she posed the question of whether one was the enlargement of the other, hoping that students would then figure the problem out. As she found Carey, one of the students who presented subtraction argument from the previous discussion, still stood by her argument, the teacher pressed the class by providing an explanation that the increase was in fact not the same in length and width across the rectangles if the original measurements were considered.

Using the illustrations as examples, the teacher intended to challenge students' conviction with the subtraction arguments, which was confirmed with her post-interview. She provided an argument that was parallel to the subtraction arguments in that both length and the width were increased or reduced by the same amount. Furthermore, she made it both "simple" and "extreme". She chose a "simple" example likely to remove unnecessary distractions from the original problem wherein the quantities were not nice to work with to focus on pattern or relationship. Also, the example she chose was "extreme" in the sense that the difference between the length and the width was huge enough to make obvious the disproportional increase in length and width across the rectangles when the same amount was added. The teacher's move to provide an extreme argument that is a special case of the main argument is what Polya calls *specialization*, one of heuristic strategies in problem solving. He explains that the use of simple examples can be a stepping stone to approach a problem when students do not know how to begin.

On the other hand, it turned out the illustrations she drew did not reflect such disproportional increase. One thing the teacher seemed to miss during the instruction was that the additive strategy by itself was not the only issue in student argumentation. Students still seemed to depend on their perceptions to make their arguments. Because the illustrations conformed to their conviction about the subtraction arguments, they did not seem to be cognitively disturbed with the teacher's example. Therefore, even though she explained how disproportionally the lengths were increased, students were hardly challenged. During the post-interview, the teacher pointed out that her illustration was not accurate enough to reflect the distortion produced from the subtraction strategy.

> I realized the one image, the sketch that I drew. I should have drawn this one, um, yeah. These just weren't proportional either, and I know that led to some problems…I should have made it much longer. And so I realize that wasn't a good thing. And I was just hoping someone was going to say, no, wait this is wrong, but then one girl did, but she did it in a way that it wasn't a big revelation. She was just really quiet about it. And the rest of the class didn't seem very convinced.

Choosing examples and illustrations is a kind of instructional practices that are related to teacher's mathematical knowledge for teaching (Hill et al., 2008). In this case, teachers' MKT was revealed in her formulation of an extreme example that was parallel to the main argument but exaggerated reasoning or faulty reasoning.

However, her exclusive focus on adding or subtracting the same amount with the extreme example did not seem helpful enough to advance students' current understanding. For one, her misrepresentation with the illustration made it hard to challenge student's perception-based reasoning. Her inaccurate drawings did not show disproportional increase properly so that ended up confirming the students' conviction instead of indicating that subtracting the same amount would produce enlargement.

Also, adding the same amount does not always produce distortion. If the same amount is added to each side of a square, it will still produce enlarged figures. So talking only about subtracting the same amount in itself was incomplete in determining enlargement. There should have been discussion about how big the increase was, by 1 in this example, compared to each measurement. For example, in one side, there was a 100% increase, a ratio of 1 to 2, whereas on the other side, there was only a 1% increase, a ratio of 100 to 101. Comparing quantities in terms of operations did not seem to make a convincing argument about disproportionality to students. The missing discussion about ratios with the extreme example might have been evidence of her limited understanding of how to disprove proportionality using important ideas in proportionality.

In sum, in responding to student difficulties, teachers either simplified arguments or used counter-arguments as an attempt to help students construct or refute arguments. Teachers used the heuristic method of decomposition to reduce cognitive complexity to help students start from easy arguments. Also, teachers used the heuristic method of using counter-arguments either by inviting students' counter-argument or providing extreme examples to challenge students' current understanding. So in responding to student difficulties, teachers used their knowledge of heuristics. Their knowledge of key ideas on proportionality was evident in their use of heuristics such as formulating sub-argument or extreme examples. Also, dealing with opposing arguments required more than teachers' knowledge of noticing the core element of underlying reasoning of student arguments because presence of opposing arguments did not always bring disequilibrium in current understanding while opening up a possibility. The teacher's limited understanding of how to strengthen counter-arguments to make them more convincing to students was observed. The findings are summarized in Table 4.4. In the following section, I discuss the third move that MKT appears to be in action in creating an opportunity for mathematical argumentation.

Table 4.4

Responding to Difficulties

Teaching Moves		Purposes	MKT demonstrated in teaching moves	MKT
Responding to Difficulties	Decompose	Simplify	Congruence as a subset of similarity Square as a subset of rectangle	• Knowledge of a concept in its relation to a subset of concept
	Use counter-arguments	Challenge	Key ideas underlying additive strategy Disproportional increase due to additive strategy exaggerated in extreme case	• Knowledge of key ideas underlying student reasoning including pre-conceptions or proportionality • (lack of) Knowledge of how to strengthen counter-arguments to challenge incorrect arguments, including formulation of extreme case

Pressing for Generalization in Defining

A third teaching move that the teacher's MKT was in action in support of classroom argumentation was pressing for generalization in defining. Pressing for generalization in general was not as frequently observed as the other two moves in all three classrooms. It is surprising given the results from MKT that showed all teachers were capable of making generalized arguments themselves in some way, they did not press students for to make generalizations where opportunities to do so existed. In the places where the opportunities existed, the teachers tended to accept empirical arguments without further pressing for generalization.

Despite its rare occurrence, it is worth noting where this move appeared in activities and how teachers supported student participation. Pressing for generalization appeared when classes made transitions from discussing copy machine enlargements to discussing textbook definitions of similarity, which I call *pressing for generalization in defining*. In two classrooms, teachers created opportunities for students to engage in mathematical argumentation to build a mathematical relationship for enlarged figures by generalizing patterns from their observations in the activity Copy Machine Enlargement. In contrast, one teacher provided a mathematical relationship for enlarged figures, which lacked a conceptual connection between what students observed and the mathematical relationship she provided.

The transition from enlargement activity to textbook definitions of similarity had to do with creating a class definition of similarity. By generalizing the patterns students observed with enlarged figures, they built a mathematical relationship for enlarged figures, which would later be used as a foundation for the class definition of similarity. Understanding of how a class definition was established in the classroom was important in the understanding of classroom argumentation because definition was deeply related to how ideas were validated. In a classroom, a definition was created out of the context that its concept arose from and was validated by students. In another, a definition was delivered from the teacher to be simply accepted as fact. An understanding of how definition as warrant was established in a classroom was also important in determining where the authority of argument resided. Because warrant is an element that has to do with giving authority to an argument, how warrant was previously established and accepted as fact mattered in determining whether the argument was external or not. It means that an understanding of the nature of an argument required understanding of how knowledge is built within a class's unique history. Therefore, understanding the establishment of class definition was important as a precursor to argumentation.

MKT was evident in the teacher's decisions to create opportunities for argumentation about a generalized relationship built from observations of enlarged figures. The decision required teacher's knowledge of two areas. One area was what could and needed to be checked for its mathematical validity. This knowledge is parallel to "knowledge of situations for proving", which is identified as teacher's MKT for proof (Stylianides & Ball, 2009). The teacher's knowledge of situations for argumentation seemed to be in use to create opportunities for argumentation in building a general mathematical relationship for enlarged figures. Another area of teacher knowledge had to do with mathematization. The teacher's decision to provide opportunities for students to build a mathematical relationship from the patterns detected from visual observation in real world context seemed to reflect the teacher's knowledge of horizontal mathematization, which is a mechanism to turn real world problems into mathematical ones (Rasmussen et al., 2005).

In contrast, the teacher's failure to provide opportunities for students to build a mathematical relationship by building on students' observations with enlarged figures implied the teacher's limited understanding of the two areas. The teacher seemed to take definitions from the textbook or her own knowledge as fact and considered it to have authority over the arguments that used the definitions as warrant. In the following sections I elaborate on these claims.

Pressing for Generalization in Defining: To Justify

One way of pressing for generalization in defining was to create opportunities for students to engage in argumentation that generalizes a relationship from the patterns students observed in a specific context where a mathematical concept is embedded. By inviting students to make general claims about statements and justify them from the context where concept of similarity was embedded, teachers provided opportunities for students to create a class definition, which later would be used as a warrant to show similar rectangles.

The following episode illustrates ways in which a teacher pressed for generalized arguments about how to figure out whether a pair of figures was a copy machine enlargement. It began with the very last part of whole class discussion of argumentation in Nancy's classroom on whether and why a pair of figures was an enlargement or a distortion. The beginning of the episode illustrates the nature of argumentation and the kinds of interaction between the teacher and students that appeared in the discussion prior to this episode.

T:	So, Ashley. Do you think there is a distortion or just an enlargement?
Ashley:	I think there's a distortion because he's shorter there.
T:	Okay, because he's shorter here? He seems squatter *[sic]* than he does here somehow. Okay. Does everyone agree that there's a distortion here? No, Joey? What do you think?
Joey:	I think it's an enlargement.
T:	An enlargement? Do you have any idea how we'd know for sure?
J:	No.
T:	No? David, can you share with us the idea that you had about this?
David:	Okay. Um. I think it was wrong, but okay. So I measured the larger gecko from tongue to tail-tip.
T:	Okay, you said you measured the long gecko.

<omitted – the teacher collected measurements both from David and other students, making sure everyone agreed with the measurements>

<On board>

Lg	T-T	8.3 cm	Sm	T-T	4.9 cm
	Knees	4.1 cm			2.5 cm

T:	Well so, if that's the case, how would we know if these two are exactly, if one is just a blow up of the other? If this is a blow up of this one? How would we know that? Carey?
Carey:	You subtract the smaller measurements from the larger measurements, and see if it's the same amount.

<omitted>

T:	Are you all okay with this? So if we subtract this from this *<indicates on the left "leg" side subtracting the 4.1 from 8.3>*, and this from this *<indicates on the right "sm" side subtracting the 2.5*

from 4.9>? We should get the same amounts? Are you all okay with this? Anybody think that might not be right?

After Ashley's argument and Joey's counter claim, the teacher sought advanced justifications (Line 8) beyond perceptual arguments typified as Ashley's argument (Line 3). To press for mathematically advanced arguments, she invited David (Line 10) to share his idea with the class, who she knew from the small group discussion had built a quantitative relationship.

At the beginning of the whole class discussion, there was an incidence where a student attempted to give a justification using measurements. However, the teacher did not pursue the student's justification but redirected her to look for details in the visual appearances of the figures. She continued to invite multiple students to provide their justifications that were basically visual in nature and accepted their justification as valid. Therefore, it was possibly the teacher's intention to give students opportunities to build their arguments mainly from visual images they were familiar with from their everyday experiences.

As David started presenting his claim, the teacher made a whole class task measuring the lengths and the widths of the figures. The omitted part from the transcript had to do with the whole class discussion of collecting measurements. By making measuring a public task, she made sure all students had the same measurements. In doing so, she limited the possibility of disagreement in students' argument that could arise due to disagreement in quantities. Eliminating such possibility can help students focus exclusively on arguing over the relationship among the quantities instead of arguing over the quantities themselves. Selecting David's claim as an object of public argumentation and eliminating a possibility for disagreement can be considered as a "filtering" move Sherin (2002) discussed. According to her, "filtering" is selecting student strategies for public discussion for their importance in a teacher's learning goals. Here, the teacher selected David's claim as an object for whole class argumentation to advance from perceptually-based to mathematical arguments. Also, by eliminating the possibility of argumentation due to information insignificant to the concept of enlargement, she helped students focus on the critical aspect.

After collecting measurements from the students, she opened up the whole class discussion by asking how to figure out whether there was enlargement (Lines 17-19). This move of pressing for generalized arguments can be considered her attempt to press her students to build a *general* claim about a *mathematical* relationship in enlarged figures. She seemed to intend for two things with this move. For one, this move can be considered as her attempt to create opportunity for student to *mathematize* the problem from real world context into a mathematical one, which is called "horizontal mathematization" (Rasmussen et al., 2005). With this move as well as the move of inviting David to share his claim, she intended to shift from basic arguments to mathematical ones. By creating opportunities for argumentation, she engaged students in building a mathematical relationship by mathematizing observations with enlarged figures.

Also, this move can be considered as her attempt to establish a *generalized* warrant. She pursued claims about quantitative relationships built from a context where a concept of similarity was embedded. These claims, when generalized, would become a warrant according to Toulmin's scheme (1958/2003), presented in Figure 4.2.

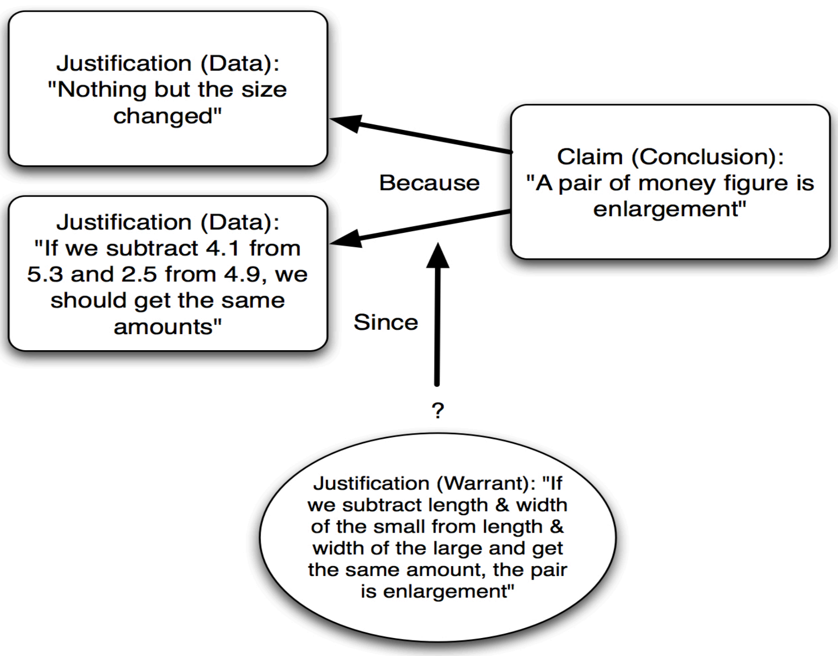


Figure 4.2. Structure of an hypothetical argument

In other words, she was looking for a generalized claim about quantitative relationships with enlarged figures, which would become a foundation for a class definition of similarity. The claims students made about the quantitative relationships for enlarged figures were challenged for their validity as the teacher invited counter-arguments (Lines 23-27). Therefore, these moves created an opportunity for students to engage in argumentation in establishing a mathematical warrant from the context of enlargement where concept of similarity arose, which later became a legitimate warrant to justify similarity.

This is not to say that a class definition can be invented by students from scratch. I am arguing that we need a more flexible view of mathematical definition. Students can engage in creating a class definition by engaging argumentation about a mathematical statement for definition built from a specific context the concept arises. They can authorize a validity of argument that uses the definition as a warrant that is built through argumentation with consensus of participants who are involved in the process in a community. This definition can then be used as a fact for later argumentation, an accepted statement to be used without challenge for its legitimacy. As a consequence, students become the ones who have the authority to determine the validity of argument by checking if an arguer uses previously shared knowledge in a community.

When students engage in making arguments over warrants, they get to engage in higher level of mathematical reasoning (Weber et al., 2008). By engaging in argumentation over warrants, students had a chance to reexamine their understanding of mathematical concepts or reasoning underlying their arguments.

After this episode, students engaged in developing a class definition of similar rectangles. The teacher introduced definitions of similar figures from the textbook and shared them in connection to mathematical relationships built from enlargement. The definitions created in this classroom involved a concept of, within-ratio or between-ratio with the use of a common multiplier, which is a key concept of similarity (Lamon, 2007). Cross-multiplication was also brought up by a student and used as a legitimate method.

In her move to press for generalization in defining, her use of MKT became evident in two areas. Her understanding of what could and needed to be checked for its mathematical validity appeared in her decision to press for generalized arguments. She saw a possibility and a need to create a mathematical definition through argumentation. The fact that she invited counter-arguments against the generalized statements about the relationship among measurements with enlarged figures, she knew the statements could and needed to be questioned for their validity.

Also, the move revealed her knowledge of how definition can be created through argumentation from the context where its key ideas are embedded. By giving opportunities for students to build a generalized relationship with enlarged figures through argumentation, she supported students to build a class definition of similarity by focusing on important key ideas. In this support appeared her understanding of creating mathematical definition from context its concept is embedded and generalizing it to become a class definition of similarity and legitimate warrant for following argumentation.

This knowledge is related to *knowledge of situations for proving*, one of MKT identified as teacher's MKT for proof (Stylianides & Ball, 2008, p. 311). According to

Stylianides and Ball, it refers to knowing how to "identify situations in which proof is called for, recognize important mathematical differences among these situations, and stage appropriate opportunities for their students to engage in proving" (p. 311). The teacher seemed to use this knowledge in her moves to turn students' mathematization of enlargement statements to become an object of argumentation.

Pressing for Generalization in Defining: To Provide

Another way of pressing for generalization was to *provide* a general relationship for students to accept it as fact, without connecting to a specific context where a mathematical concept arises. Regardless of the context its concept arose from, a teacher provided a general relationship among measurements with enlarged figures.

The following episode illustrates how a teacher shared a mathematical relationship with enlarged figures, which would later become a foundation for a class definition of similarity. The clip came from whole class discussion of whether a pair of money figures were copy machine enlargements or not and why. Prior to this point, student justifications were mostly perceptual even if the teacher, Susie, kept pushing students to focus on length and width. Right before this conversation, Kate claimed that one was a shrunk version of the other, but she was not successful in providing a proper reason. Then Kevin volunteered to provide his observation opposing Kate's claim.

Kevin: Smaller one is <*inaudible*>

T: Okay it looks like one is shrunk and one is- <*approaching to Kevin, showing laminated sheet closer*> is either shrunk or enlarged?

Kevin: This one is longer though <*pointing the smaller Nicaraguan bill, horizontally with a ruler*>

T: Is smaller one longer than the top?

Kevin: The smaller one is longer though. Right there it's shorter <*pointing to the larger Nicaraguan, vertically*> that one <*smaller one*> compared to the one <*bigger one*> <*inaudible*>

T: Okay everybody what I'd like you to do is- Kate. Everybody eyes on me. Kevin is saying that the sizes are NOT proportional. <*pointing with hands both bills, with fingers widened*> So, what I'd like you to do is <*one hand up*> Come on. Excuse me. Using your ruler, measure the two-dollar bills. <*Nicaraguan money*> Measure the length <*pointing horizontal side*> and the width <*pointing vertical side*> and see if they are proportional.

Kevin provided his observation about the side lengths in the smaller object in comparison with the larger one as an objection to Kate's claim. While his claim was not explicitly stated in his argument, he raised a doubt on Kate's claim that the pair was enlargement by starting his statement by saying "this one is longer though" by referring to a side length in the smaller object. Kevin's participation became an entry to an opposing argument, where Kevin provided a justification for why the pair was not an enlargement.

When Kevin stated his argument, the interaction between Kevin and the teacher remained rather private between the two. Kevin explained by pointing out referents on the sheet of paper that was held by the teacher standing next to him in such as way that the conversation between the two could not really be shared with the others.

However, when rebroadcasting Kevin's counter-argument to the whole class, the teacher expanded on his statement by making his informal description into a more formal one by inserting a technical term. Kevin's justification is more advanced than the justifications that previously appeared, which were based on visual images such as "looks squatter *[sic]*". Instead of visual images, he used the lengths of the sides to justify distortion though description. While his observation suggested he saw disproportionality among measurements, it is not clear the connection between Kevin's argument and the teacher's interpretation of Kevin's argument made sense to both Kevin and the other students. While parts of what Kevin said appeared to be inaudible from video, the field note indicated that he did not use the term proportional but the teacher introduced the term as she addressed Kevin's argument to the whole class.

As one can see in the following episode, after the teacher introduced the term "proportional", she immediately made a transition to work on proportion equation. After Kevin made the argument against it being a copy machine enlargement, the teacher asked students in groups to figure out the measurements of the pair of money and discussed them as a whole class. She then posed the question of how to figure out with the measurements if the pair was a case of enlargement.

T: Okay, now then. Shh. We are saying they are copies. How do we know they are copies? <*pause 5 sec*> Okay, can we look at the proportion? Can we set this as proportion? Proportion.

Lg	**Width = 4 in**
	Length = $3\frac{1}{2}$
Sm	**Width = $3\frac{1}{2}$**
	Length = $2\frac{1}{2}$

Michelle: Yes.
T: Can we say, what would be the proportion then?
Michelle: 4 over 3.5?
T: what does that mean? Are we doing the width of the large over, what would be the bottom? *<writing on OHP>*
Michelle: Length of the large.
T: Length of the large. That should be equal to what then?
Michelle: 4 over 3.5.
T: 4 over 3.5 and what's about the other ratio then?
S: Width of the small.
T: Width of the small over.
S: Length of the small.
T: Length of the small. So what are the length, so what are the numbers then?
S: 3.5.
T: 3.5 over?
S: 2.5.

$$\text{Proportion} = \frac{Wid\ lg}{Lg\ lg} = \frac{Wid\ sm}{Lg\ sm}$$
$$= \frac{4}{3.5} = \frac{3.5}{2.5}$$

T: 2.5 so. If, shh… we are saying that those two are machine copies. Before we can conclude whether that is correct or not, we are gonna look at similar figures. Those are all similar figures, and I want you to get the definition of what similar figures are.

Soon after the class agreed on measurements, the teacher posed a question of how to figure out if the pair of Nicaraguan money was a case of enlargement (Lines 1-2). Just like the previous teacher, her question could have been an opened up to argumentation. As discussed previously, the transition from Kevin's argument and the teachers' interpretation of Kevin's argument was a mathematically critical juncture in terms of both mathematization and generalizing observations to establish a class definition of similarity. This was a place students had the opportunity to construct a mathematical relationship through argumentation using their thinking skills where a mathematical concept was embedded. Furthermore, this was a place where a class definition of similar rectangles would be founded.

However, she made a different move at this juncture from the other two teachers. She suggested that students use proportion. Instead of asking students questions to press further to mathematize and generalize Kevin's observation, she asked students to set up a proportion (Lines 2-3). In the previous small group and whole class discussions on figuring out whether a pair of money figures were copy machine enlargements or not, she kept pushing students to use lengths and width. Also, right before this episode she rephrased Kevin's argument by inserting the term "proportional" and asked students to find proportion. Therefore, when she posed the question on how to figure out if a pair of Nicaraguan money was a copy machine enlargement, she was likely to have only one response in mind, justification using proportion.

Then she followed up with a series of questions to elicit information that was needed to set up a proportion (Lines 6-23). She asked fill-in-the-blank questions to bring out the information, ratios, to fill in the proportion. She asked questions in the order that the proportion had to be set up. She first asked for one ratio and then for another one to put the other side of the equation, while writing the equation on board. Instead of helping students to bring out the concept of ratios from the context of enlargement in comparing measurements, either within or between measurements, she brought up the term ratio as part of setting up the proportion (Line 13). The concepts such as proportion and ratio seemed to stand alone as terms without their meanings in connection to enlargement.

Also, she asked questions about what a number in each ratio represents. With these questions, she seemed to intend to make sure students set up a proportion correctly. It is a common error that students set up a proportion incorrectly by putting ratios that do not correspond. In various places in the activities, the teacher had emphasized the concept of correspondence. She used the term as “a key word” and clarified it with examples when discussing meaning of the definitions of similar rectangles. She even provided a warm-up task where students had to practice finding corresponding sides in a few pairs of rectangles. Therefore, her knowledge of student error seems to come into play in asking question to elicit student knowledge of what numbers in proportion represent to ensure that students succeed in solving problems of proportionality in correctly setting up proportions.

The interaction between teacher and students appeared to be a “funnel” interaction in that she guided students with a series of questions to model her thinking process of solving proportion (Wood, 1994). Through funnel interaction, she guided students to set up a proportion correctly. By following the guidance, students were able to set up a proportion successfully. However, the reasoning of why proportion could set up the way it was to represent the proportional relationship with enlarged figures was left unexplained. The reasoning remained implicit in teacher’s guidance.

After this episode, she asked students to cross-multiply with the proportion that they discussed. She did not discuss why students had to cross-multiply or why it was a legitimate way to show enlargement or proportionality. After calculation, she simply gave an explanation shown in the following episode.

> T: You guys. The point is, the point is- the point is when you cross-multiply, they are not equal. So they are not proportional <*writing inequality sign between 10 and 12.25*> they are not proportional. So they are not actually, even though it looks like those two are similar, Eddie. Even though they are not, they look like they are similar, they are not similar. Because the size are not proportional. So using that knowledge, with your partner, can you come up with a similar rectangles now.

Cross-multiplication has no immediate connection to concept of enlargement or similarity. It is a common method to show proportion. It is a property that has an "if-and-only-if" relation to proportion. It is an efficient method to check for equivalence in a proportion, especially when the quantities are so messy that no obvious relationships are detected. However, the method is hard to make any sense of in the context of enlargement or similarity because it only entails a mathematical property that does not having meaning within the context. This is not to say that cross-multiplication should not be taught in learning similarity. The teacher needs to recognize that cross-multiplication is accepted as a legitimate justification for proportionality by students because of her authority.

Susie's MKT was evident in her sequence of moves in making transition from enlargement to defining similarity. Her acceptance of external argument was evident in her move to insert a technical term and to make the transition to quantitative arguments. Without explaining how the idea of proportion could be built from Kevin's argument proposed from the context of enlargement, she simply introduced the rule of proportion to students. Kevin's argument could have been a seed from which key ideas of proportion could have sprouted. However, she did not provide an opportunity for students to mathematize Kevin's argument, but provided a way that did not make sense in the context of enlargement. While the justification she gave was mathematically valid and generalized to show similarity from an objective sense, it was a justification that was unreasonable in the context of enlargement. Therefore, she seemed to accept an external justification as valid even if it lacked meaning created through understanding.

This knowledge became more apparent in later discussion of the textbook definitions. Susie delivered textbook definitions in a manner that ensured students would understand what each parts of definitions mean. Then, she picked a textbook definition of "having equivalent ratios" and proceduralized it as cross-multiplying or simplifying. She did not explain how the definition and the procedures were related to enlargement or similarity. Students simply had to accept them as definitions or facts and apply them in their justification to show similarity, which the teacher then accepted as valid.

This possibly lent to her limited knowledge of different aspects of mathematical definition. Her acceptance of cross-multiplication without questioning it indicated she did not seem to recognize cross-multiplication did not embody concept of enlargement or similarity. While it is a mathematically valid justification to show similarity, cross-multiplication did not entail key ideas of similarity, such as how both length and width should increase to create enlarged or similar figures—idea using between-ratio—or how the shape for each figure is kept—idea using within-ratio (Lamon, 2007). She did not seem to realize a statement with if-and-only-if-relationship to a concept definition might not carry key ideas of the concept.

In her sequence of moves to help students work out proportion, her knowledge of textbook definition and student errors in applying the definition was evident. In her guidance, she made sure students correctly set up a proportion by clarifying what each number in proportion represented in relation to measurements of the two figures. She seemed to use the definition of similarity from a textbook, which stated "corresponding sides have the equivalent ratios", in her guidance. She first guided students to set up proportion and then how to cross-multiply to show equivalence. Using her knowledge of precise definition from the textbook and common student errors, she gave clear guidance of how to apply the definition in problem solving.

Her clear guidance seemed to support student argumentation in the two activities, Finding Your Similar Rectangle and Making Similar Rectangles. In group presentations for both activities, no mistakes were observed in the students' use of cross-multiplication and simplification to justify that their pair of rectangles were similar. However, while students were able to solve similarity problems, whether students actually used key ideas for proportion and similarity in particular was unclear.

In sum, teachers made different decisions over how to press for generalization in defining. One decision was to create opportunities for students to engage in argumentation in building a mathematical relationship for enlarged figures by observing patterns from a specific context. In contrast, the other decision was to provide a mathematical relationship between the enlarged figures without making connections to the context. This decision became a foundation for a class definition of similarity, which in turn affected the nature of argumentation that used the definition as warrant. In the former, the teacher's MKT was observed in her noticing which situation to turn into argumentation. In the latter, the teacher's limited knowledge of how a mathematical definition can be formulated was evident in her limited way of establishing definition. The findings are summarized in Table 4.5.

Table 4.5

Pressing for Generalization

Teaching Moves	Purposes	MKT demonstrated in teaching moves	MKT
Pressing for Generalization In Defining	Justify	A general mathematical relationship among measurements in similar figures Justification of the relationship	• Knowledge of what can and needs to be justified • Knowledge of constructing a mathematical definition from the context its concept is embedded
	Provide	(lack of) mathematical connection between concept and procedures How to set up a proportion Correspondence of measurements	• Knowledge of student error in applying procedures • (lack of) Knowledge of concept definition and proceduralized definition

SUMMARY

In this chapter, I discussed the three themes for teaching moves and their purposes emerged from cross-case analysis of transcripts from three case teachers. The three moves in which teacher's use of MKT was evident were Revoicing by Reformulation, Responding to Difficulties, and Pressing for Generalization in Defining. In each move, I discussed teacher's MKT that was demonstrated in the moves to support classroom argumentation such as teacher's knowledge of core elements of argument and heuristics. Also, teacher's knowledge of how class definitions could be created from the context it arises was discussed. The findings are summarized in Table 4.6. In the following chapter, I discuss implications and limitations of the study.

Table 4.6

Summary of Findings

Teaching Moves	How teaching moves are enacted (examples)	What is accomplished (purposes)	What mathematics is demonstrated in teaching moves	Mathematical Knowledge for Teaching
Revoicing by Reformulation	Repeat or rephrase student statement by making implicit information explicit. e.g. Rephrase "They both are one half" to "So they have the same ratios."	Highlight core information	Key ideas for proportional reasoning such as "having equivalent ratios", within- or between-ratios Key ideas underlying student conceptions such as "adding the same amount to both sides" for additive reasoning	Procedures such as cross-multiplication or simplification to show equivalence of ratios Knowledge of key ideas of proportional relationships Knowledge of student preconception
	Rephrase student statement by adding new information that is not extracted from student statement. e.g. Rephrase "They are equal. They both are half." to "So you simplified and get one half".	Introduce new information	Previously shared ideas such as definitions or procedures	Knowledge of how to validate ideas (lack of) Knowledge of making connections between student statement and teacher reformulation

Table 4.6 (cont.)

Responding to Difficulties	Decompose an original problem into simpler ones. e.g. Change a question of similarity to a question of congruence	Simplify	Congruence as a subset of similarity Square as a subset of rectangle	Knowledge of a concept or definition in its relation to a subset of concept
	Use counter-arguments. e.g. Deal with counter-arguments or give an extreme case of counter-arguments by highlighting critical features underlying counter-argument	Challenge	Key ideas underlying additive strategy Disproportional increase due to additive strategy exaggerated in extreme case	Knowledge of key ideas underlying student reasoning including pre-conceptions (lack of) Knowledge of how to strengthen counter-arguments to challenge incorrect arguments, including formulation of extreme case

Table 4.6 (cont.)

Pressing for Generalization	Help students gain conviction for generalizability by collecting multiple arguments or multiple evidence. e.g. "Does this pattern work with other pairs?"	Justify	A general mathematical relationship among measurements in similar figures Justification of the relationship	Knowledge of what can and needs to be justified Knowledge of constructing a mathematical definition from the context its concept is embedded
	Provide a general method. e.g. Tell students to use "proportion"	Provide	(lack of) mathematical connection between concept and procedures How to set up a proportion Correspondence of measurements	Knowledge of student error in applying procedures (lack of) Knowledge of concept definition and proceduralized definition

CHAPTER FIVE: DISCUSSION

SUMMARY AND IMPLICATIONS

Mathematical argumentation is essential to performing mathematics. While there is heightened awareness of the importance of promoting mathematical argumentation in teaching and learning mathematics in all grade levels, we as a field have only begun to understand what and how mathematical knowledge for teaching is used in instructional practices in general and argumentation in particular (Hill et al., 2008). This study explored teachers' knowledge-in-action using a case study with three embedded cases of teachers with high MKT. The study demonstrated how these teachers used MKT in teaching moves to support student engagement in constructing and advancing mathematical arguments.

Teachers used a variety of moves to support student argumentation. These moves included revoicing, responding to student difficulties by invoking heuristic cases, and generalizing patterns by establishing connections. The teachers' use of revoicing for reformulation to explicate student reasoning was consistent with findings from other studies (O'Connor & Michaels, 1996; Forman et al., 1998). In revoicing, teachers used reformulation to highlight core information in student statements and to introduce new information she expected to come up in student statements. When student difficulties arose, teachers provided heuristic arguments that included simplified or extreme cases or counter-arguments to help students advance their arguments. Findings also indicated that not all case teachers provided opportunities for students to engage in argumentation during development of mathematical definitions, which had a significant impact on the nature of the argument that followed.

The study also identified MKT that was used to support argumentation. Teachers needed to notice core mathematical features such as the warrant underlying a student's argument in order to make it an object of whole-group examination for their mathematical validity. When student difficulties arose, instead of giving direct evaluation or explanation, teachers sometimes used heuristic arguments to help students construct or challenge arguments. Teachers used knowledge of decomposition to reformulate an original argument by breaking it into sub-arguments. Decomposition also involved knowing about the relationship between the original and sub-argument. At other times, teachers used knowledge of special cases to reformulate an original argument to highlight the type of reasoning underlying student difficulties. In both cases, teachers' knowledge of heuristic argumentation was evident in their reformulation of arguments. In dealing with opposing arguments, teachers needed to choose a counter-argument that could disprove a given argument by addressing the cognitive conflict between the two. Furthermore, teachers' knowledge of what could be and what needed to be validated with generalizations was indicated by their decision of when to create an opportunity for student argumentation. A teacher's knowledge of a mathematical definition and its key ideas was invoked when she created an opportunity for students to use argumentation to establish a definition.

The study was designed on the belief that MKT would be closely related to teaching practices, so it chose to identify MKT-in-action that appeared in teaching moves. However, teaching moves are affected by a variety of other factors such as teacher beliefs, learning goals, or curriculum tasks (Hill et al., 2008). Despite the teachers' ability to understand generalized arguments to some degree indicated in the MKT assessment, the study found very few occurrences of generalized arguments in their classrooms. This could have been affected by the teachers' beliefs about students as well as teachers' limited understanding of how to support students in making generalizations. Also, I speculated that one teacher's struggle with management issues was a contributing factor in the kind of argumentation her teaching practices supported. This teacher expressed frustration because she felt students did not behave well. This was the classroom in which external arguments appeared most frequently. Both her dependence on external arguments that was evident in this teacher's MKT assessment and her issues with maintaining control of the classroom could have contributed to this tendency.

This study's findings have implications for both the theory and practice of teaching. First, by adopting an emergent perspective on mathematical proof and argumentation, the study provides an alternative perspective on classroom practices of mathematics. Mathematical practice is distinguished from other practices for its deductive system. However, deductive arguments are not the only way in which mathematical knowledge is developed; heuristic arguments are also used, especially in creating and advancing ideas (Lakatos, 1976; Polya, 1945). Therefore, it is important for students to learn to engage in creating heuristic as well as deductive arguments. By building on this broad perspective, the study investigated how ideas were developed and accepted as true through argumentation in middle grade classrooms.

Second, the study helps us think about reformed teaching in a different way. Numerous researchers have attempted to identify reformed classroom practices by focusing on promoting conceptual understanding or mathematical communication. However, teaching practices that support student participation in authentic mathematical discourse such as argumentation have not yet been well characterized (Franke, Kazemi, & Battey, 2009). This study adds to the efforts to identify teaching practices that support mathematical argumentation in classrooms.

Third, there have been a few attempts to identify MKT for general teaching (e.g. Learning Mathematics for Teaching project by the University of Michigan) or with specific content (e.g. Knowledge of Algebra for Teaching by the Michigan State University). As a complement to these efforts, the current study advances the field by focusing specifically on argumentation practices. Prior research on MKT showed, for example, the importance of the teacher's knowledge of mathematical language, which included use of mathematically precise and pedagogically comprehensible definitions (Hill et al., 2008). This study contributes to understanding teachers' knowledge of how mathematical definitions could be created, refined through, and used in argumentation by students. Not only did teachers need knowledge of mathematical definitions, but they also had to understand how a definition could be reformulated in the context where its concept is embedded and accepted its use as legitimate in argumentation.

Furthermore, this study extends existing research on teachers' knowledge of proof. There have been numerous studies that have investigated teachers' knowledge of argumentation from a limited perspective, that of argumentation as proof. These studies investigated the teacher's understanding of deductive schemes in determining the validity of arguments. However, adopting an emergent perspective on mathematical proof and argumentation, as this study did, shows the need to broaden our perspective with respect to MKT for proof and argumentation. Further, most of these studies used surveys or interviews to identify the teachers' knowledge rather than rich analysis of classroom practices; these analyses allowed the present study to introduce and explore the idea of MKT-in-action, which explicates how teachers' knowledge is used in practice to support student engagement.

The findings also have implications for practice. First, the study has implications for curriculum development. Emergent perspectives on mathematical argumentation suggest the importance of student engagement in mathematical argumentation from the early grades on. Because argumentation is essential to developing knowledge of mathematics, which includes both heuristic and deductive schemes, students in all grades need to learn how to participate in argumentation. However, the current curriculum in most schools focuses on a limited notion of mathematical argumentation as a separate topic, a way of determining mathematical validity. Consistent with this observation, the present study showed that teachers used argumentation mainly to validate whether rectangles were similar, and not much to advance ideas for the concept of similarity. Development of proportional reasoning could have been supported if teachers had provided more opportunities for students to engage in argumentation not only to validate but also to develop ideas for similarity. In general, the curriculum needs to be more specific about argumentation as an integrated process in developing concepts.

Second, the study has implications for teacher learning. Teacher support was critical to students' use of argumentation to validate ideas. However, because of limitations in teachers' knowledge of how argumentation could be used to develop concepts and how concepts could be used to advance argumentation, teachers' support for students was limited at times. High content knowledge alone does not appear to be sufficient to support the development of student argumentation. Therefore, what a teacher should learn in preservice education and professional development needs to be revisited. For one, teachers' knowledge of mathematical argumentation needs to be extended to include emergent perspectives on mathematical argumentation and heuristic approaches beyond the usual deductive approach. The teachers' knowledge of mathematical definition, for example, needs to go beyond knowing a precise definition and delivering it to students in a clear manner. Teachers need to also know how to unpack a definition so that it can be reformulated in a specific context in which its concept arises and used in support of argumentation.

Along with reconsidering teacher learning, creating MKT assessments that address teacher's understanding of argumentation is needed. In addition to teachers' understanding of topics, MKT assessment should also address the teacher's knowledge of mathematical practices such as argumentation and how mathematical concepts and ideas are advanced through the practices. For example, a possible assessment item might ask teachers to formulate various examples to challenge a student argument that is incomplete or only partially correct. This kind of item requires knowledge of students' current reasoning and different role and use of examples.

LIMITATIONS

This study was limited by the scope of its data. Because the study focused on argumentation in the context of whole class discussion, for a small number of teachers teaching a replacement unit that lasted only a few days, it did not produce enough data to test the strength or generalizability of some of the findings. For example, pressing for generalization in establishing class definitions only occurred once for two of the teachers. While there was a pattern suggested by those two episodes as discussed in Chapter 4, more data would have been helpful to confirm or refute the finding. Another way of pressing for generalization that was observed in one teacher could not at all be checked for pattern matching within the case or between other cases. While it was considered a negative example of the pattern that appeared in two teachers, more data would have been helpful in confirming it as indicative of a pattern in and of itself – and in fact, my observations of many other classrooms that were not part of this dissertation study suggested that this single episode was not unique but actually very common.

Another limitation of the study was a result of the limited information held in the transcripts of each teacher's instruction. There are two reasons for this limitation. For one, because of classroom noise and overlapping speech, some important data was missing and prevented a full picture of knowledge building through argumentation. For example, some parts of student utterances were missing, so it made it difficult to determine whether a teacher's uptake was a simple repetition of what a student had said or involved significant expansion. Also, because whole class discussion was primarily of interest, small group discussions were only partially transcribed. While field notes were made to get a general impression of teacher interaction with students during small group discussion, more data from these discussions would have been helpful in understanding how interaction between the teacher and students for developing argumentation.

The study was also limited in investigating argumentation from Lakatos's notion of the "zig-zag" approach of conjecturing and justifying statements. There was limited opportunity for students to construct conjectures, partly due to teacher's limited use of curriculum and the amount of time assigned for the unit. Most arguments that appeared in the study involved choosing between two claims, either similar or not similar, instead of constructing a conjecture such as "if you subtract the smaller measurements from the larger measurements and get the same amount, a pair of figures is similar". Therefore, the findings were bound to be limited in studying classroom argumentation and teaching moves and MKT that supported argumentation because of limited use of argumentation in the classrooms under investigation.

DIRECTIONS FOR FUTURE RESEARCH

Findings from this study suggest several possible directions for future research. One is to investigate teachers' knowledge that was used to advance arguments. As illustrated in the teaching move Responding to Difficulties, teachers used heuristic argumentation strategies such as examining special cases and decomposing into simpler arguments when student difficulties arose. These heuristic strategies were meant to challenge at an appropriate level or scaffold beginning steps instead of directly evaluating or correcting students' mistakes and misconceptions. Knowledge for advanced argumentation addresses two important learning goals. One is that with teacher support, students can learn to use heuristic strategies when advancing their own arguments. Students as well as teachers need to understand the role of heuristic arguments in advancing ideas, by recognizing the current status of argument and coming up with heuristic arguments to further refine the original argument as discussed in Lakatos's notion of development of ideas through a zig-zag path between conjecturing and refuting. Another important point is that this kind of knowledge is important in giving an entry point for students to grapple with creating arguments without telling or evaluating.

Another direction is to investigate the moments during instruction when teachers see a need to engage students in argumentation, as was discussed in the move Pressing for Generalization in Defining. As indicated in the discussion of arguments and their nature, argumentation mostly occurred as prescribed in the curriculum in the activity of justifying the similarity of rectangles and was an important learning goal of the replacement unit. However, argumentation did not appear as frequently in the activity of defining. Not all teachers saw a need to create opportunities for argumentation. In particular, when students were reasoning on the quantitative relationships among the measurements of two figures was the moment where the key idea of proportional reasoning could come up, which was relationship between the two relationships. Two of the teachers saw a need to create opportunities for students to engage in argumentation to build this key idea whereas one teacher did not. As a consequence in classrooms in which this opportunity was created, student preconceptions based on additive reasoning were made visible and could be challenged through argumentation. On the other hand, in the classroom where this opportunity was not created, no argument appeared that used an additive strategy and this possible student preconception was left unexamined. Future research could explore where in the development of ideas a teacher sees a need for argumentation and what impact it has on student learning.

CONCLUSION

This study was designed to identify teaching moves that supported argumentation and how the teacher's mathematical knowledge for teaching was used in these moves. Teaching moves such as revoicing by reformulation, responding to difficulties, and pressing for generalization in defining were useful sites for investigating teacher's MKT. The findings suggest that supporting argumentation require MKT that is more sophisticated than MKT for general purposes. Knowing precise and comprehensible definitions is not enough for a teacher to be able to build it through argumentation. Furthermore, knowing how to construct a counter-argument is not enough for a teacher to be able to use it to create a space for cognitive conflicts to arise.

However, high MKT does not ensure productive argumentation, although it is a necessary factor. Other factors may also contribute to productive classroom argumentation. One is classroom management. One teacher struggled with student engagement and management of the class. Her struggle could have contributed to her level of support of students' exclusive dependence on external arguments, in which authority for mathematical validity was placed in teacher or textbooks. Another has to do with curriculum. Teachers seemed to rely on the curriculum material in their support of argumentation because arguments occurred as prescribed in the material. Because teachers used a replacement unit provided by the research team for classroom observation, they tended to stick to curriculum material.

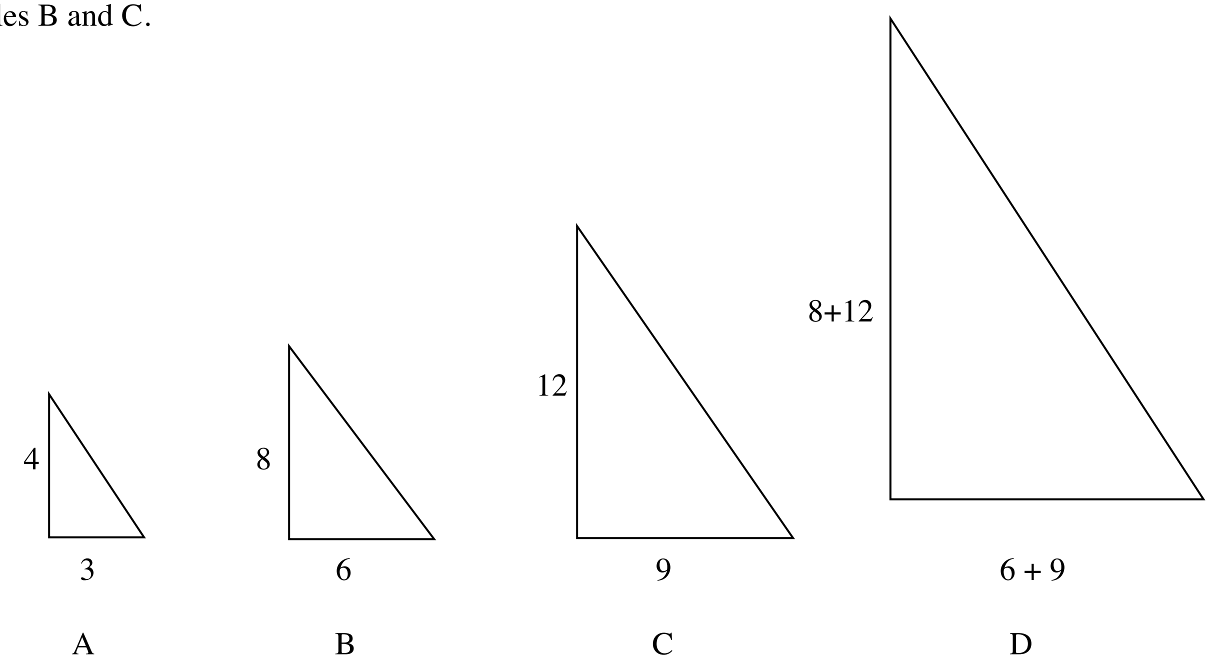
The study findings highlight that orchestrating classroom argumentation is a challenging practice that requires sophisticated knowledge of mathematics for teaching as well as the use of a variety of teaching moves to support student participation in argumentation. In classrooms where multiple challenges were present such as students' lack of experience with mathematical argumentation, students' lack of verbal participation, or issues with classroom management, just to name a few, more sophisticated support from the teacher would be necessary to engage students in argumentation. For example, teachers need to constantly work on establishing sociomathematical and social norms for student participation in classroom discourse

(Yackel, 2002). More research is needed to identify teaching moves that support mathematical argumentation and to identify MKT for argumentation. Are there other teaching moves or other classroom practices that are effective in supporting mathematical argumentation? What are other areas of MKT that are critical in support of argumentation? Are they different from MKT for general purposes and how? How should heuristic and deductive arguments be used in classroom instruction to support argumentation in middle grades? What do they look like in collective argumentation and how teacher can support it? There is much left to be answered.

APPENDIX A. Example Items of MKT Assessment

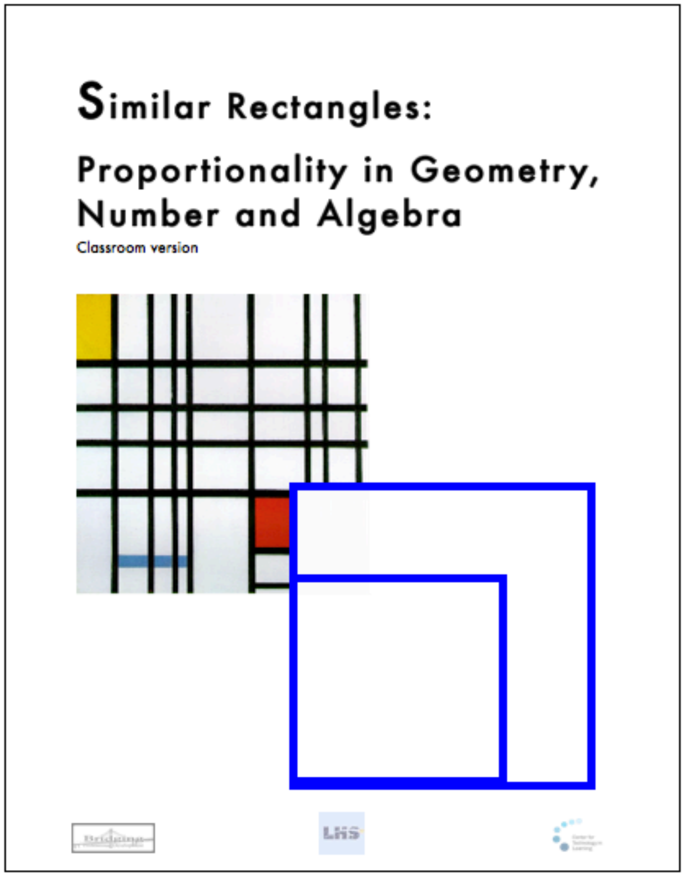
ITEM 4. Write a precise definition of “similar polygons” that middle school students could understand.

ITEM 6. Students are asked to generate triangles similar to a right triangle A as shown below. Most students generated similar triangles by multiplying each side by the same number. Jose, however, generated a triangle, D, by adding sides of two other similar triangles B and C.



Does this strategy work in general for right triangles?

APPENDIX B. Similar Rectangles



Copy machine Enlargement to Similarity

1. What do you know about "similar figures" in geometry? Discuss. Then, find the definition of similar figures in your textbook, and write it below.

2. Discuss the relationship between copy-machine enlargements and similar figures. Then, with your class, agree on a definition of similar rectangles that you will use (there are many). Write it below.

Find Your Similar Rectangle

1. Find the rectangle that is similar to yours.

2. Write 2 different explanations of why your rectangles are similar. The explanations should be mathematically different. (P.S., what is your conjecture?)

Making Similar Rectangles

1. Make a set of at least 5 rectangles, all similar to each other. Use more than one method for making them. Use different tools!

2. Describe your methods for creating your similar rectangles.

3. Now we have a conjecture:

All the rectangles in this set are similar to each other.

Justify this conjecture, two different ways.

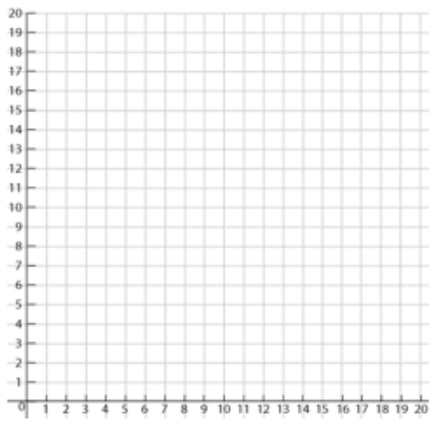
4. Find at least 3 patterns that work for *all 5 of your* rectangles.

   a. Describe the patterns.

   b. Make another rectangle similar to yours. Choose one pattern. Does the pattern apply to it?

   c. Can you make a rectangle that is similar to your 5 that does not follow the pattern? Either make it or explain why you can't.

5. Record the lengths and widths of your rectangles on the graph and table below. Use the graph to find another pattern—one that will work for all rectangles similar to yours. That's your conjecture; now explain why you think it is true.



What is your pattern?

6. List the properties of similar rectangles.

If two rectangles are similar, we know that…

References

Anderson, R., Chinn, C., Chang, J., Waggoner, M., & Yi, H. (1997). On the logical integrity of children's arguments. *Cognition and Instruction*, *15*(2), 135-167.

Bakhtin, M. M. (1981). *The dialogic imagination: four essays*. Austin, TX: University of Texas Press.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Math, Teachers, and Children* (pp. 216-235). London: Hodder & Stoughton.

Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A. J. Bishop (Ed.), *Mathematical knowledge: Its growth through teaching* (pp. 175-192). Dordrecht: Kluwer.

Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Yearbook of the national society for the study of education, Constructivism in education* (pp. 193–224). Chicago, IL: University of Chicago Press.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3–14). Edmonton, AB: CMESG/GDEDM.

Ball, D., Lubienski, S. T., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of Research on Teaching* (4th ed., pp. 433–456). Washington, DC: American Educational Research Association.

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. *Educational Studies in Mathematics*, *11*, 23-29.

Boero, P., Chiappini, G., Garuti, R., & Sibilla, A. (1995). Towards statements and proofs in elementary arithmetic: An exploratory study about the role of teachers and the behavior of students. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th Psychology in Mathematics Education Conference*, (Vol. 3, pp. 129-136). Recife, Brazil: Universidade Federal de Pernambuco.

Boero, P., Garuti, R., & Mariotti, M. (1996). Some dynamic mental processes underlying producing and proving conjectures. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 121-128). Valencia, Spain: Universitat de Valencia.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School*. Portsmouth, NJ: Heinemann.

Cazden, B. C. (2001). *Classroom discourse: The language of teaching and learning*. Portsmouth NH: Heinemann.

Chazan, D. (1993). High school geometry students' justifications for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, *24*(4), 359-387.

Chazan, D. & Ball, D. L. (1999). Beyond being told not to tell. *For the Learning of Mathematics*, *19*(2), 2-10.

Chinn, C. A., & Anderson, R. C. (1998). The structure of discussions that promote reasoning. *Teachers College Record, 100*(2), 315–368.

Civil, M., & Planas, N. (2004). Participation in the mathematics classroom: Does every student have a voice? *For the Learning of Mathematics, 24*(1), 7-12.

Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics*, *23*, 99–122.

Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis, 12*, 311–330.

Common Core State Standards Initiative (2010). *Common Core State Standards for Mathematics*. Unknown: Author.

de Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras*, *24*, 17-24.

Erduran, S., Simon, S., & Osborne, J. (2004). Tapping into argumentation: Developments in application of Toulmin's argument pattern for studying science discourse. *Science Education, 88*, 915–933.

Fennema, E., Franke, M. L., Carpenter, T. P., & Carey, D. A. (1993). Using children's mathematical knowledge in instruction. *American Educational Research Journal, 30*(3), 403–434.

Forman, E. A., Larreamendy-Joerns, J., Stein, M. K., & Brown, C. A. (1998). “You’re going to want to find out which and prove it”: Collective argumentation in a mathematics classroom. *Learning and Instruction, 8*(6), 527–548.

Franke, M. L., Kazemi, E., & Battey, D. (2009). Understanding teaching and classroom practice in mathematics. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 225-256). Charlotte, NC: Information Age Publishing.

Hanna, G. (1991). Mathematical Proof. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 54-61). Boston: Kluwer Academic Publishers.

Harel, G., & Sowder, L. (1998). Students’ proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education III* (pp. 234-283). Providence, RI: American Mathematical Society.

Healy, L. & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education, 31*(4) 396-428.

Heaton, R. (1992). Who is minding the mathematics content? A case study of a fifth-grade teacher. *Elementary School Journal, 93*(2), 153–162.

Herbst, P. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education, 33*, 176-203.

Heritage, J. (1984). A change-of-state token and aspects of its sequential placement. In J. M. Atkinson & J. Heritage (Eds.), *Structures of social action: Studies in conversational analysis* (pp. 299-345). Cambridge, UK: Cambridge University Press.

Hersh, R. (1990). Proving is convincing and explaining. *Educational Studies in Mathematics, 24*, 389-399.

Hersh, R. (1997). *What is mathematics, really?* Oxford, UK: Oxford University Press.

Hiebert, J., & Stigler, J. (2000). A proposal for improving classroom teaching: Lessons from the TIMSS video study. *The Elementary School Journal, 101*, 3-20.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California’s Mathematics Professional Development Institutes. *Journal for Research in Mathematics Education, 35*, 330–351.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*, 371–406.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory story. *Cognition and Instruction, 26*, 430-451.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal, 102*, 59-80.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 229-269). Hillsdale, NJ: Lawrence Erlbaum Associates.

Knuth, E. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*, 379-405.

Krussel, L., Edwards, B., & Springer, G. (2004). The teacher's discourse moves: A framework for analyzing discourse in mathematics classrooms. *School Science and Mathematics, 104*(7), 307- 307-312.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge, UK: Cambridge University Press.

Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 629-668). Charlotte, NC: Information Age Publishing.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*(1), 29-63.

Lampert, M. (2001). *Teaching Problems and the Problems in Teaching*. New Haven, CT: Yale University Press.

Lampert, M., Rittenhouse, P., & Crumbaugh, C. (1996). Agreeing to disagree: Developing sociable mathematical discourse. In D. R. Olson & N. Torrance (Eds.), *The handbook of education and human development: New models of learning, teaching, and schooling* (pp. 731-764). Oxford, UK: Blackwell Publishers, Inc.

Lehrer, R., & Franke, M. (1992). Applying personal construct psychology to the student of teacher's knowledge of fraction. *Journal for Research in Mathematics Education*, *23*(3), 223-241.

Lehrer, R., & Lesh, R. (2003). Mathematics learning. In W. Reynolds & G. Miller (Eds.), *Comprehensive Handbook of Psychology*, (Vol. 7, pp. 357-391). New York, NY: John Wiley.

Loncoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Newbury Park, CA: Sage Publication.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Maher, C. & Martino, A. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27*(2), 194-214.

Marshall, C., & Rossman, G. (2006). *Designing qualitative research* (4th ed.). Thousands Oaks, CA: Sage Publication.

Martino, A. & Maher, C. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior, 18*(1), 53-78.

Mehan, H. (1985). The structure of classroom discourse. In T. A. Van Dijk (Ed.), *Handbook of discourse analysis* (Vol. 3, pp. 119-131). London, UK: Academic Press.

Mercer, N. (1995). *Guiding the construction of knowledge: Talk amongst teachers and learners*. Clevedon, UK: Multilingual Matters, Ltd.

Mintrop, M. (2004). *Schools on probation: How accountability works (and doesn't work)*. New York, NY: Teachers College Press.

Nathan, M., & Kim, S. (2009). Regulation of teacher elicitations in the mathematics classrooms. *Cognition and Instruction*, *27*(2), 91-120.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

O'Connor, M. & Michaels, S. (1996). Shifting participant frameworks: orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning and schooling* (pp.63-103). New York: Cambridge University Press.

O'Connor, M. C. (1998). Language socialization in the mathematics classroom: Discourse practices and mathematical thinking. In M. Lampert & M. Blunk (Eds.), *Talking mathematics* (pp. 17–55). New York: Cambridge University Press.

Polya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Polya, G. (2004). *How to Solve it*: A new aspect of mathematical method. Princeton, NJ: Princeton University Press.

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advanced mathematical activity: A view of advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*, 51-73.

Schoenfeld, A. (1994). *Mathematical thinking and problem solving*. Hillsdale, NJ: Erbalum.

Schoenfeld, A. (1985). *Mathematical problem solving*. New York, NY: Academic Press.

Schwarz, B., Neuman, Y., & Biezuner, S. (2000). Two wrongs may make a right…if they argue together! *Cognition and Instruction*, *18*(4), 461-494.

Shechtman, N., Roschelle, J., Haertel, G., & Knudsen, J. (2010). Investigating links from teacher knowledge, to classroom practice, to student learning in the instructional system of the middle-school mathematics classroom. *Cognition and Instruction*, *28*, 317-359.

Sherin, M. G. (2002). A balancing act: developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, *5*, 205-233.

Shulman, S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4-14.

Shulman, S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*, 1-22.

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 15*, 3-31.

Sowder, J. T., Phillip, R. A., Armstrong, B. E., & Schappelle, B. P. (1998). *Middle-grade teachers' mathematical knowledge and its relationship to instruction*. Albany, NY: SUNY Press.

Stein, M. K., Baxter, J. A., & Leinhardt, G. (1990). Subject-matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal, 27*(4), 639–663.

Strauss, A., & Corbin, J. (2007). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Thousand Oaks, California: Sage Publication.

Strom, D., & Lehrer, R. (1999, April). The epistemology of generalization. In J. Kaput (Chair), *Toward a research base for algebra reform beginning in the early grades. Symposium conducted at the annual meeting of the American Educational Research Association*, Montreal, Canada.

Stylianides, A. J. (2006). The notion of proof in the context of elementary school mathematics. *Educational Studies in Mathematics, 65*, 1-20.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*, 289-321.

Stylianides, A. J., & Ball, D. (2008). Understanding and describing mathematical knowledge for teaching: knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11*, 307-332.

Swafford, J. O., Jones, G. A., & Thornton, C. A. (1997). Increased knowledge in geometry and instructional practice. *Journal for Research in Mathematics Education, 28*(4), 467– 483.

Thompson, P., & Thompson, A. (1994). Talking about rates conceptually, Part I: A teachers' struggle. *Journal for Research in Mathematics Education, 25*, 279–303.

Thurston, W. P. (1998). On proof and progress in mathematics. In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics: An anthology* (pp. 337–355). Princeton, NJ: Princeton University Press.

Toulmin, S. (1958). *The uses of arguments*. Cambridge: Cambridge University Press.

Toulmin, S. (2003). *The uses of arguments* (updated ed.). Cambridge: Cambridge University Press.

Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classroom. *Review of Educational Research*, *78*(3), 516-551.

Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussion: warrants become the objects of debate. *Educational Studies in Mathematics*, *68*, 247-261.

Well, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education*. Cambridge, UK: Cambridge University Press.

Wertsch, J.V. (1991). *Voices in the Mind: A Sociocultural Approach to Mediated Action*. Cambridge, MA: Harvard University Press.

Wood, T. (1994). Patterns of interaction and the culture of mathematics classroom. In S. Lerman (Ed.), *The culture of the mathematics classroom* (pp. 149-168). Dordrecht: Kluwer.

Wood, T. (1998). Funneling of focusing? Alternative patterns of communication in the mathematics class. In H. Steinbring, M. G. Bartolini-Bussi, & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 167-178). Reston, VA: National Research Council of Teachers of Mathematics.

Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education, 30*(2), 171-191.

Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education*, *37*(3), 222-255.

Yackel, E. & Cobb, P. (1995). Classroom sociomathematical norms and intellectual autonomy. In L. Meira & D. Carraher (Eds.), *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education (*Vol. 3, pp. 264-271). Recife, Brazil.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 9–24). Utrecht, Netherlands: Freudenthal Institute.

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior, 21*(4), 401–518.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477.

Yin, R. (2003). *Case study research: Design and methods* (3rd ed). Thousand Oaks, CA: Sage.